Week 3 Kinematics

In a beginning physics course, the first order of business is analyzing **kinematics.** Etymologically, kinematics derives from the Greek from ***kīnein,*** to move and is defined as *the branch of mechanics that studies the motion of a body or a system of bodies without regard to the forces that caused it.***.** In other words, if a body is in motion, and we know its position, velocity and acceleration at a given instance, we can predict what those quantities will be a minute from now, without knowing any details about the forces causing the motion. We only require that no new forces act on the body in that minute interval.

When, farther down the road, we want to decipher *how* the forces cause the motion, that branch of physics is the study of **dynamics**. The Greek for this word is ***dunamikos*** or ***dunamis***. From this same root we derive *dynamite*. It is a word that should bring to mind forces and determinant power. Meanwhile, let's define ***displacement***, ***velocity*** and ***acceleration,*** in turn::

displacement—a change in position when a body moves from an initial position to a final one,

$\Delta\vec{x} = \vec{x}_f - \vec{x}_i$. Notice that a displacement is a vector quantity. Both a magnitude and direction should properly be specified. By contrast, distance is a scalar quantity, it has no direction attached to it. When you drive to pick up your nephew from the day care 1 kilometer away and return straiight home, your displacement is 0, but the distance traveled is 2 km.

Example. Mark leaves his home and walks two blocks north on Elm street to visit his friend Paul. He then departs from Paul's house and walks five bocks south to visit Joe.

(a) What total distance did Mark walk in blocks ? So far Mark has walked 7 blocks.

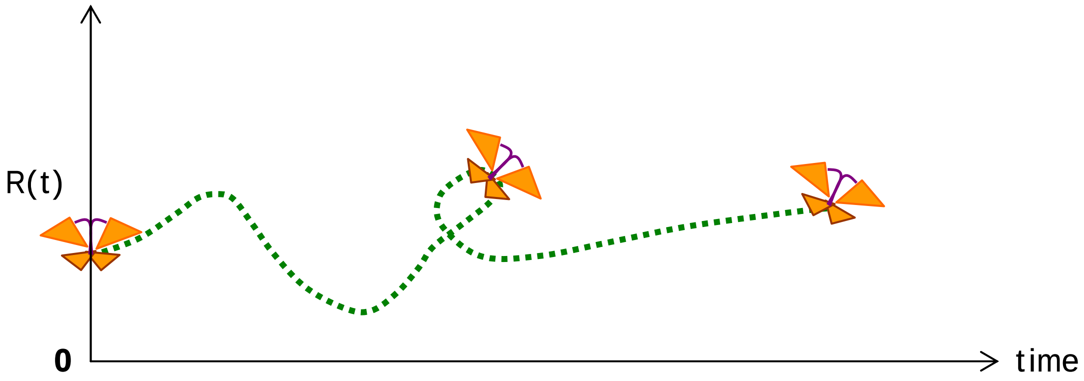
(b) What was Mark's net displacement in blocks? must give direction. 3 blocks south

(c) If Mark returns straight home what would be his displacement from Joe's house?
3 blocks north

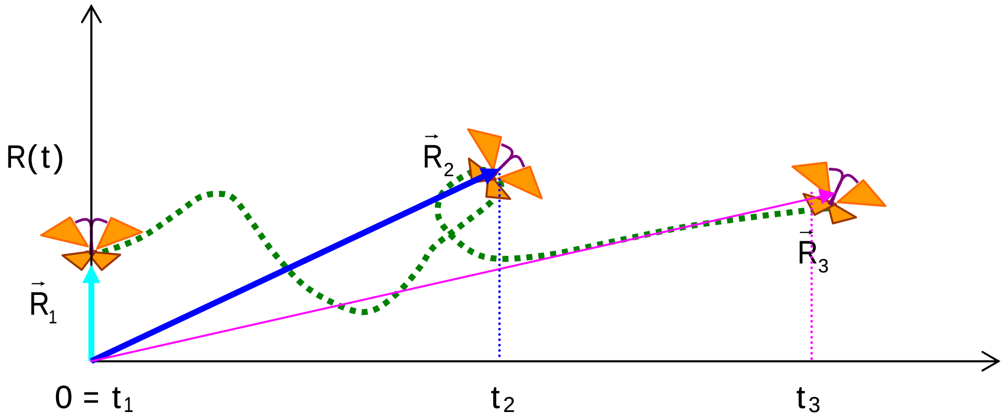
(d) If (c) occurred, now what total distance had Mark walked? 10 blocks

(e) If (c) occurred, now what was Mark's net displacement? 0 block.
So consider the meandering flight of a lazy butterfly :

We can use the path that the butterfly traced out to think about displacement: Let's capture that path in a **reference frame**—a **coordinate system** that allows us to measure position relative to origin of the reference frame versus time.

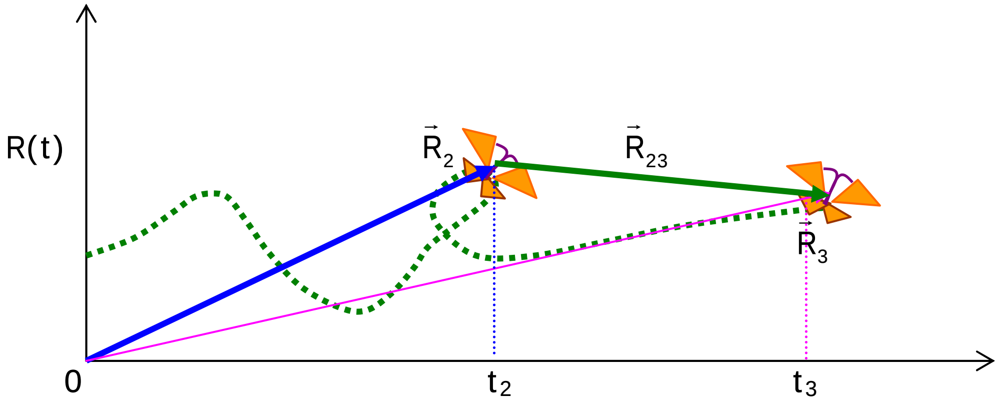


Next, draw position vectors $\vec{R}_1$, etc., that point from the origin to where the butterfly is:

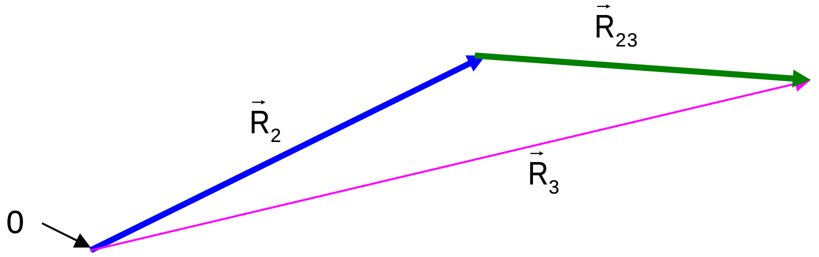


We say that the butterfly is at position $\vec{R}_1$ at t_1, $\vec{R}_2$ at t_2, etc.. In the vicinity of $\vec{R}_2$, the butterfly moves in a loop and makes no net progress. Eventually it flits on to $\vec{R}_3$,

We're now in a position to think about displacement. What is the net change in position in moving, say from $\vec{R}_2$ to $\vec{R}_3$? In other words, what is the butterfly's displacement ?



Graphically, the displacement $\vec{R}_{23}$ is just an arrow drawn from the arrowhead of $\vec{R}_2$ to the arrowhead of $\vec{R}_3$. Mathematically we have: $\vec{R}_2 + \vec{R}_{23} = \vec{R}_3$

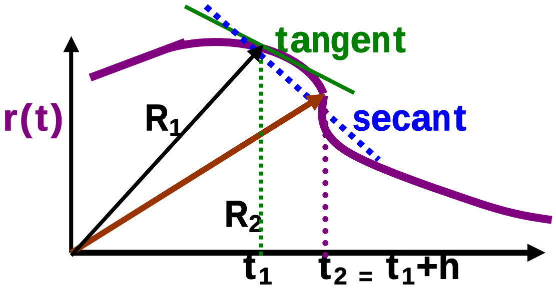


If you started at the origin (the base of $\vec{R}_2$) and took a stroll along the **blue** and the **green** path to its end (its **arrowhead)**, you'd arrive at the same point as if you had walked along the **pink** path, $\vec{R}_3$. This is essentially how vector addition translates.

Now since the butterfly arrived at $\vec{R}_3$ from $\vec{R}_2$ in a time interval $\Delta t = t_3 - t_2$, you could also determine the **average velocity** of the butterfly during that time interval:

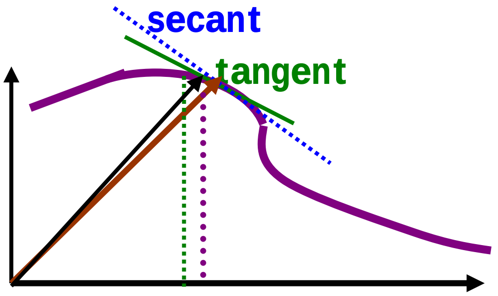
$$\bar{v}_{AVE} = \frac{\vec{R}_3 - \vec{R}_2}{t_3 - t_2} = \frac{\vec{R}_{32}}{\Delta t_{32}}$$

So now, consider a more general displacement curve, that displaces the position of an object at any given time and as a collective is just the object's trajectory. The displacement vectors $\vec{R}_1$ and $\vec{R}_2$ point to the object's position at t_1 and t_2 respectively. The average slope of the curve is graphically the dashed segment—called the **secant**-- and computationally:



$$\textbf{ave. slope} = \frac{R(t_2) - R(t_1)}{t_2 - t_1} = \frac{R(t_1 + h) - R(t_1)}{h}$$

The slope of the secant differs noticeably from that of the true tangent. This quantity represents the *average velocity* in the time interval from t_1 to $t_2 = t_1+h$.



Next let h approach zero—the secant (dotted) approaches the instantaneous slope (the solid segment), and mathematically:

$$\text{Lim }(h\to 0)\ \frac{R(t_1 + h) - R(t_1)}{h} \equiv \frac{dR}{dt} \equiv \textbf{velocity}\ \ \mathbf{v}$$

In other words, the instantaneous slope, the **tangent**, of the displacement curve at time *t* is just the velocity of the object at that instant.

A similar development for a plot of velocity vs. time would show that the instantaneous slope of the velocity curve at a point is the acceleration of the object at that time.

$$\text{Lim }(h\to 0)\frac{v(t_1+h)-v(t_1)}{h} \equiv \frac{dv}{dt} \equiv \textbf{acceleration, a}$$

So far, we have considered taking derivatives (finding the tangents of displacement or velocity curves). Let us now consider the physical significance of the area under velocity or acceleration curve.

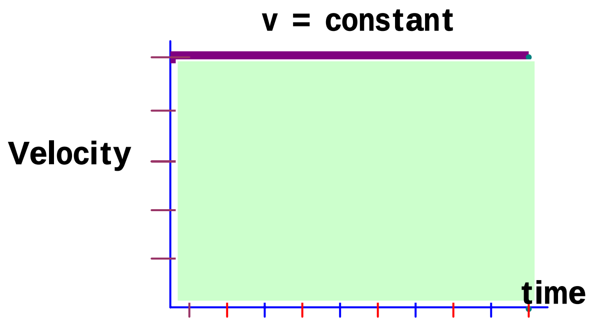
We commence with the kinematics of constant velocity ($a = 0$), then consider the case of constant acceleration ($a = \text{constant}$.)

Velocity constant

From traveling experience, we know that speed ×time = distance covered. Formally: $v\Delta t = \Delta s$, where Δs is the displacement. As by definition,

$\Delta s = s - s_o$ (s_o initial position), and $\Delta t = t - 0 = t$ we arrive at

$a = 0$ $\quad v = \text{const}$ $\quad s = s_o + vt$

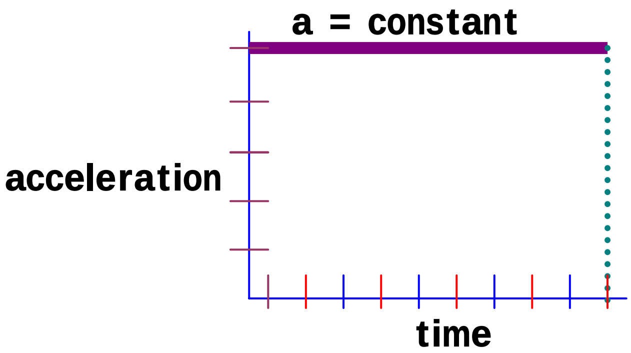


Notice, if velocity curve is above the time axis, as in this case, the area under the curve is interpreted as positive displacement Otherwise, if the curve is below, (-) negative displacement.

In two dimensions (xy plane):

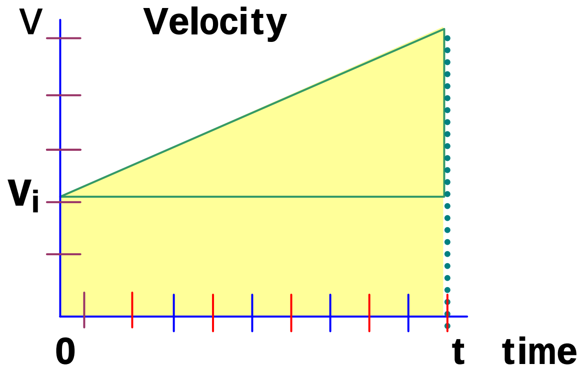
$a = 0 \quad \vec{v} = v_{ox}\hat{i} + v_{oy}\hat{j} = \text{constant} \quad x = x_o + v_{ox}t \quad y = y_o + v_{oy}t$

We restrict our attention to uniformly accelerated motions for now, thus a plot of acceleration vs. time looks like:



What would signify constant acceleration on a velocity vs. time plot?

Well, as the tangent of the velocity function corresponds to acceleration, acceleration is constant, the tangent/slope of the velocity function must be cor Velocity must increase or decrease at a set rate:



Let v_i be the initial velocity at time 0 and v be the velocity at time t. The area under the straight line is ▭ minus ◢

The first term, ▭ is the displacement if there had been no acceleration.

The 2nd term , ◢ is the additional displacement due to acceleration. Algebraically:

$$[1]\ \text{Area} = v_it + \frac{1}{2}(v - v_i)t = \frac{1}{2}(v + v_i)t$$

the problem is, usually we know the initial velocity and the acceleration, but we *don't* know v at time t. So we need to substitute for v in terms of v_i, a and t, quantities that we *do* know. From the equation of a straight line, $v = v_i + at$. Substituting into [1], we have:

$$[2]\ \text{Area} = \frac{1}{2}(v + v_i)t = \frac{1}{2}(v_i + at + v_i)t = \frac{1}{2}at^2 + v_it$$

However, as the area under a velocity curve equals displacement,

$$[3]\ \Delta s = s - s_o = v_it + \frac{1}{2}at^2$$

Another useful kinematic relation corroborates initial and final velocities, constant acceleration and displacement. For instance, a vehicle is travelling 13.3 m/s (30 mi/hr) and sees a cute little bunny rabbit in the road 15 m away.

The driver slams on the brakes, (a = 10.1 ms^{-2}). Will the vehicle miss the rabbit?

To answer questions like this, we go back to the kinematics.

Once again, the area under the curve is Area $= \frac{1}{2}(v + v_i)t$

but this time we substitute for time using the straight line equation for velocity: $t = (v-v_i)/a$ So saying we arrive at [4]:

$$[4]\ \text{Area} = \frac{1}{2}(v + v_i)t = \frac{1}{2}(v + v_i)(v-v_i)/a = \boxed{\frac{1}{2a}(v^2-v_i^2) = \Delta s}$$

Examples

(4) A ball is tossed straight up on the moon (a = 1.62 ms^{-2}) with an initial speed 3.12 m/s from a height of 3.0 m above the surface.

(a) find the time (s) at which speed of ball is zero

Solution : Maximum height is reached when v vanishes.

Find time from $v = 0 = v_i + at = 3.12 - 1.62t$.

Solving **$\underline{t^* = 1.92\ s}$** (Why the minus sign for acceleration?)

(b) Find the maximum height (m) reached by the ball

Solution When the ball stops ascending it has reached its maximum height. We found the tme for this to occur in (a). Now substitute into $\mathbf{s = s_i + v_i t + \frac{1}{2}at^2}$ where $s_i = 3.0$ m, $v_i = 3.12$ m/s

$a = -1.62$ ms^{-2} (why the minus sign?)

Thus $s = 3.0 + 3.12t^* - 0.81t^{*2} =$ **$\underline{6.00\ m}$**

(c) Find the time (s) at which ball hits the ground.

This occurs when s = 0. So we solve for

$0 = 3.0 + 3.12t - 0.81t^2$ Using the quadratic formula, we obtain,

$$\mathbf{t} = \frac{-3.12 \pm \sqrt{(3.12)^2 - 4(3.0)(-0.81)}}{2(-0.81)} = -0.8\ s \text{ or } 4.65\ s$$

Which one do we choose? Well, we know that the ball reached its peak at 1.92 s. So it can't have reached the moon's surface *before* it reached its peak, b/c ball was thrown straight up. So we choose **4.65 s**. **Also we can't have an event that ends 0.8 s before it started**!

Extra: Why doesn't this time equal twice the time to reach the peak?

(d) What is the speed (m/s) of the ball when it hits the ground?

Solution: Using v = 3.12 – 1.62t and substituting t = 4.65 s, we obtain **v = <u>-4.43 m/s</u>** (What does the negative sign signify? Answ: ball originally directed up. Reach peak, reverses, and accelerates down. If up is (+) then down is (-))

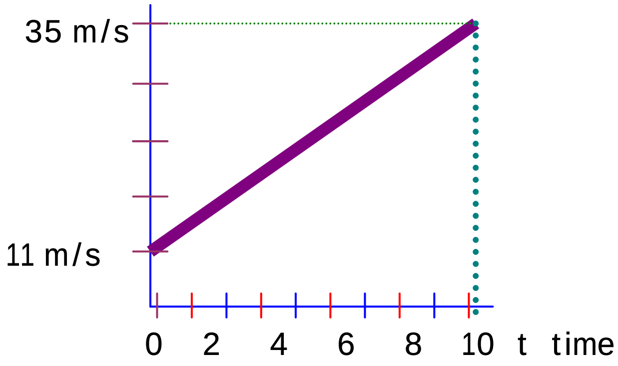
(2) A vehicle is travelling 13.3 m/s (30 mi/hr) and sees a cute little bunny rabbit in the road 15 m away. The driver slams on the brakes, ($a = 7.10\ \mathrm{ms}^{-2}$). Will the vehicle miss the rabbit?

Strategy : Given $v_i = 13.3$ m/s $v_f = 0$ (implied by completely stop) and $a = -7.10\ \mathrm{ms}^{-2}$ (why the minus sign?).

Find the displacement Δs; Compare to 15 m. If $\Delta s < 15$ m, rabbit will make it. Otherwise, not good.

As no time interval is given, seek a formula that doesn't involve time:
$(v^2 - v_i^2)/2a = \Delta s$ imply $(0 - (13.3)^2)/(2 \times 7.10) =$ **12.4 m**.
So the vehicle misses the rabbit. Yay.

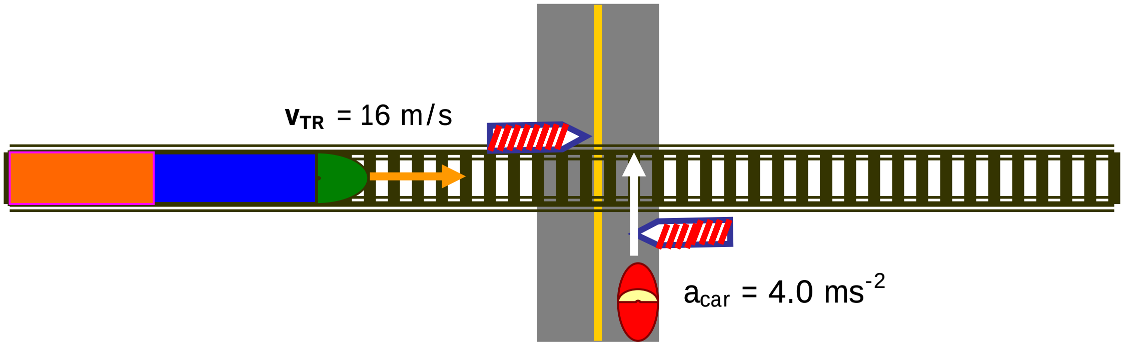
(3) A taxi accelerates from 11 m/s to 35 m/s in 10. seconds.



(a) Determine the acceleration (ms^{-2}) and (b) find the distance traveled in the 10. second interval. Solution: a = Slope of line = (35 –11)/(10-0) = **<u>2.4 ms⁻²</u>** .
Next one could use
$\Delta s = (s - s_o) = v_o t + \frac{1}{2} a t^2$ with $a = 2.4\ \mathrm{ms}^{-2}$, $v_o = 11$ m/s and t = 10 s.
Or one could use $\Delta s = (v^2 - \mathbf{v_o}^2)/2a$ Either way, one obtains **Δs = <u>230 m</u>**.

(4) A cargo train is traveling at a constant velocity of 16.0 m/s . When the train is 40.0 m from a railroad crossing, the driver of a vehicle decides to ignore the railroad sign "beat the train". The car must clear 12.0 m to get to safety. If the driver accelerates with constant acceleration $4.00\ \mathrm{ms}^{-2}$ from rest, will vehicle clear the track before the train arrives?

Solution: Sketch a drawing to get a visual picture—



—or otherwise organize train and vehicle information:

Train: v = 16.0 m/s a = 0, and we might as wel set initial position, x_o = 0 so x = vt implies $t = x/v = \dfrac{40.0\text{ m}}{16.0\text{ ms}^{-1}} = 2.50$ s.

Vehicle: $s = ½at^2 + v_ot + s_o$ becomes, with only a non-zero, $s = ½at^2$

Implies $12.0 = 0.5(4.00)t^2 = 2.00t^2$ solving t = 2.44 s ***Too close for comfort***. Any hesitation or glitch and, not good! In 2005 there were 145 vehicle train collisions resulting in 75 injuries and 72 fatalities[1]. Fortunately, most new cars have accelerations of 5.0 ms^{-2} or more.

The next example affords opportunity to show when the area under (part of) a velocity vs. time curve should be interpreted as negative. This occurs in those part of the curve where the velocity itself is negative.

(5) A subway train travels south at 5.4 m/s de-accelerates to a halt in 20.0 seconds. The train waits for 1 minute. It then reverses course, (northbound) starting from rest to reach 11.m/s in 14.0 seconds. Let the deacceleration be leg I, the wait leg II, the reverse travel leg III.

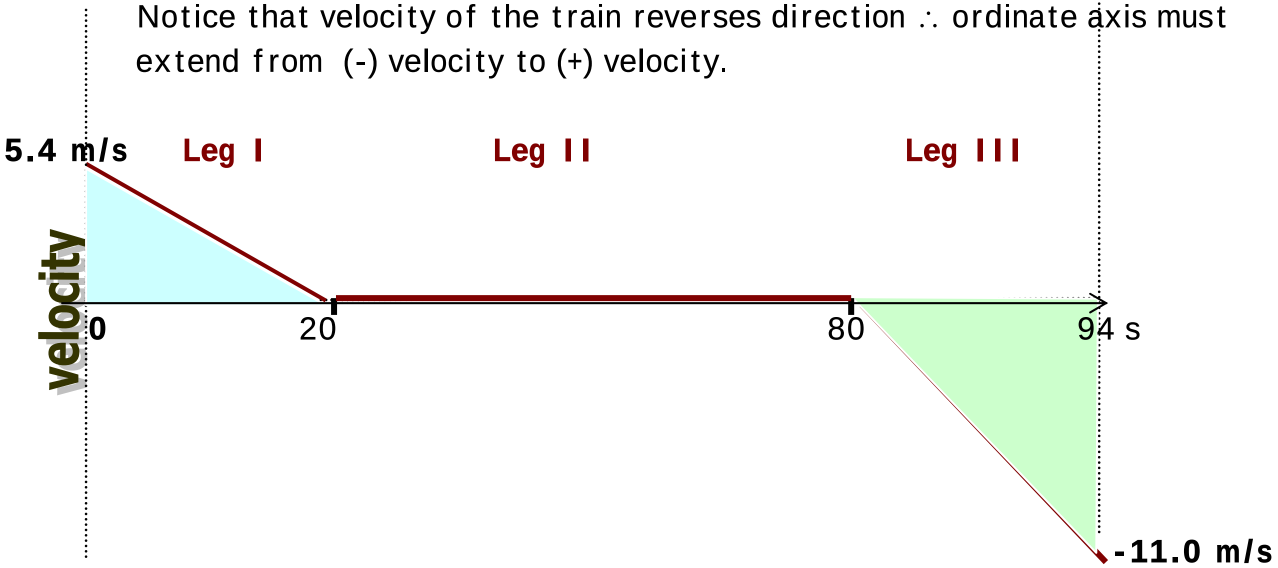
(a) Find the acceleration in leg I. If using sign, state your convention, else use cardinal direction (N or S)
(b) Find the displacement in leg I. Indicate direction as well.
(c) Determine the displacement after the first 80 seconds.
(d) Find the acceleration in leg III. Indicate direction
(e) What is the net displacement for the trip (legs I, II and III) ?

---

[1] http://www.sos.state.il.us/departments/archives/di/560__002.htm updated 6/14/2006

(f) What is the total distance traveled for the trip?

Solution. A sketch of velocity vs. time for the 3 legs goes a long way in organizing all this information as well as seeing what is being asked in (a) – (f). Notice that velocity of the train reverses direction ∴ ordinate axis must extend from (-) velocity to (+) velocity.



TIP:

Don't cramp your sketch. Make it big enough so you can see what is going on. It doesn't have to be drawn to scale but it shouldn't look like a Picasso, either. In other words, note that (i) –11 m/s is placed further South on right axis than 5.4 is placed North on left axis. (ii) The interval drawn for 60 s looks longer than the ones for 20 and 14 seconds.

(a) Let South, the original direction train was headed be positive. Then, a = (0 – 5.4) $ms^{-1}/20\ s$ = **$-0.27\ ms^{-2}$** (b) $(v_f^2 - v_o^2)/2a = -(5.4)^2/(-0.54)$ = **+54 m South**. Or area of triange = ½ (-0.27)(20) = 54 m (c) 80 seconds includes all of leg I and leg II. Thus, Δs = leg I + leg II = **still +54 m South**, b/c the train doesn't move during phase II.

(d) a = (-11–0)/14 = **$-0.79\ ms^{-2}$** ((-) b/c North). (e) Already found displacement for I and II. For leg III, it's easiest to use the triangle Area = ½ Height times base = ½ (11)(14) = 77 m.

$\Delta s_{III} = (v_f^2 - v_o^2)/2a = (11^2 - 0)/(2\times -.79) =$ **-77 m**.

Thus Δs = 54 + 0 –77 = **-23 m** or **23 m North.**

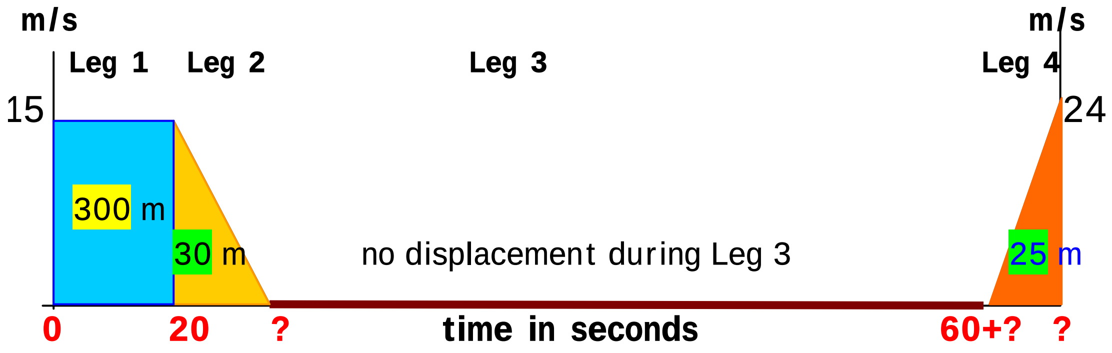
(f) Distance = 54 + 0 + |-77| = **131 m.**

(e) and (f) highlights the difference between displacement , a vector, and distance, a scalar.

(6) An extra one for *you* to try:

A train travels east 15 m/s for 20 seconds. It uniformly deaccelerates from 15.0 m/s, stopping after it has traveled 30 m. After waiting 1 minute, the train takes off again, reaching 24.0 m/s after it has traveled 25 m. **Find the time interval (s) from the beginning of the first phase to reaching 20 m/s (end of the 4th stage).**

We don't know times, but we can determine displacements from the simple areas:



Thus we may use the displacements and change in velocities to determine the time intervals in Legs 2 and 4:

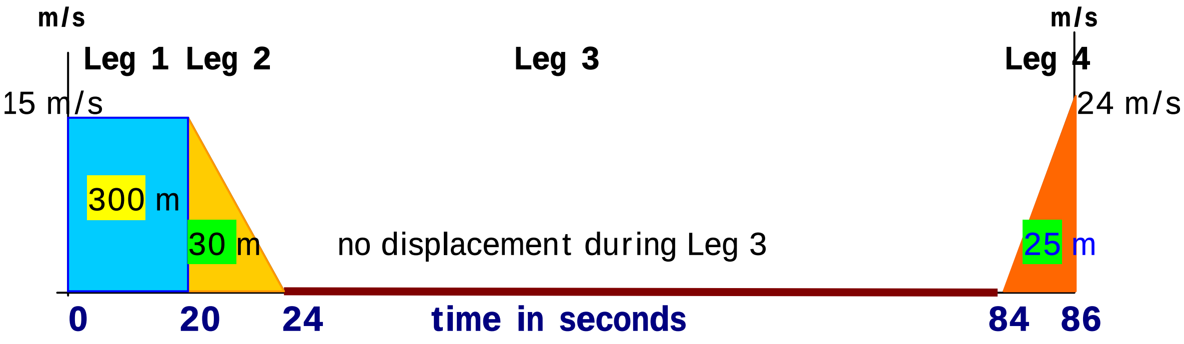
Leg I : Displacement is (15)(20) = 300 m, for your information, but this calculation was unnecessary.

Leg II: We know that the train displaced 30 m in the time that it took to stop. We can therefore solve: ½ (height) × ? = 30 that is, ½ (15 ms^{-1})× ? = 30 m. Solving, the unknown time interval ? = 30/7.5 = 4.0 s

Leg III: No displacement, hurray!

Leg IV: We know that the train displaced 24 m in the time that it took to stop. We can therefore solve: ½ (height) × ? = 30 that is, ½ (25 ms^{-1})× ? = 24 m. Solving, the unknown time interval ? = 24/12.5 = 2.08 s ~ 2.1 s.

Thus, the times become:



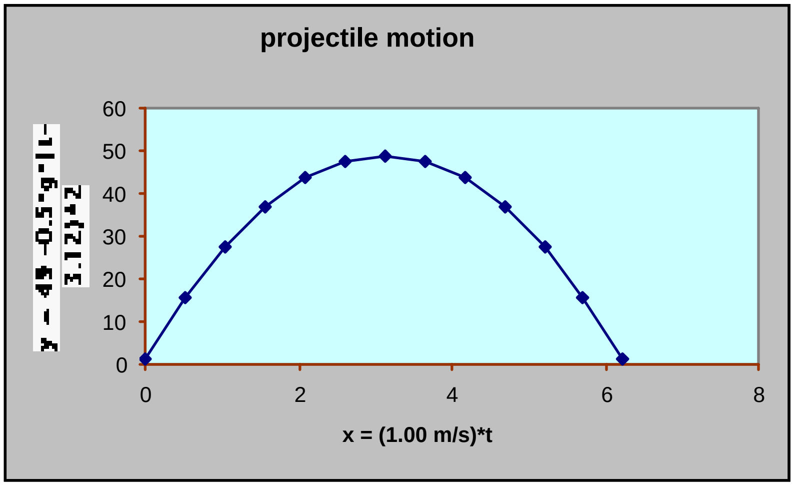
Summary (1 dimensional or quasi-1 dimensional):

Quantity	**Expression**
acceleration	a = constant
velocity	$v = (v_o + at)$
Average velocity	$v_{ave} = ½(v_o + v_f)$
position	$s = ½at^2 + v_o t + s_o$ $D = ½at^2 + v_o t$
Relation between v, v_o, a & $\Delta s = s - s_o$	$\mathbf{v^2 - v_o^2 = 2a\Delta s}$ D or Δs = **displacement**

Projectile Motion

A projectile is an object upon which the only force acting is gravity. The motion can be thought of as the sum of force-free motion in the horizontal (x) direction and vertical free fall in the -y direction. The sum of the two at any moment—vertical free fall motion and constant horizontal motion—results in a parabolic trajectory. The object will advance uniformly along the horizontal direction (same distance in each consecutive time interval) but will fall vertically a greater distance in each consecutive time interval. The graph below shows vertical vs. horizontal displacement for a projectile fired at an angle from ground level. Notice the symmetry about the max ht. (maximum height).

The next graph shows the path generated if the projectile was fired with an initial horizontal velocity from the top of a cliff.



Notice that its form looks like the right half of graph 1.

We now have enough tools to analyze this motion and make kinematic predictions.

So consider an object fired from ground level with an initial velocity $v_o = v_{ox}\,\hat{i} + v_{oy}\,\hat{j}$ where x ⇔ horizontal and y ⇔vertical. Let θ be the angle that the velocity makes with the horizontal. Usually the initial speed v_o and angle θ are given. Firstly, we have

X displacement: $x = v_{ox}t$ Y displacement $y = y_o + v_{oy}t - ½gt^2$

$= v_{oy}t - ½gt^2$

X- Velocity: $v_x = v_{ox} =$ constant Y-velocity $v_y = v_{oy} - gt$

From trigonometry, the x and y components of the initial velocity are:

$$v_{ox} = v_o\cos(\theta) \qquad v_{oy} = v_o\sin(\theta)$$

In terms of v_o and θ then, we rewrite

$$x = [v_o\cos(\theta)]t \qquad y = v_o\sin(\theta)t - ½gt^2 \quad (y_o = 0)$$

Time to achieve maximum height, maximum height and range:

Our projectile ascends against gravity ; as it ascends, its y component of velocity becomes smaller and smaller until v_y vanishes.

Then the projectile reverses its course and descends, accelerating as it does so in the y direction, but ever marking steady pace in the x direction.

The time at which y-component of velocity vanishes:

$$v_y = 0 = v_{oy} - gt, \qquad t^* = v_{oy}/g = v_o\sin(\theta)/g$$

The maximum height at this time is $y_{max} = \dfrac{(v_o\sin\theta)^2}{2g}$

The range covered when the projectile hits the ground again is:

$R = v_{ox}(2t^*)$, since in twice the time that projectile reaches maximum height, it reaches the ground again, so

$$R = 2v_{ox}t^* = 2\,v_{ox}\,v_{oy}/g = 2v_o^2\cos(\theta)\,\sin(\theta)/g = \mathbf{v_o^2\sin(2\theta)/g}$$

Notice that the horizontal distance (x) scales with time, in fact graph displayed on page 7 is parameterized such that x = 1*t. *Thus, the same graph can be interpreted as the y position of the projectile vs.* <u>*time*</u> *for the initial conditions set forth in each case.*

Examples:

7. An arrow is shot with initial speed of 20.0 m/s at an angle 60^o above horizontal. Find (a) the x and y components of its (a) velocity after 3.00 seconds (b) displacement after 3.00 seconds.

(a) $v_x = v_{ox} = v_o\cos(\theta) = (20.0\ m/s)\cos(60^o) = 10.0\ m/s$

Initial speed in the y direction is $v_{oy} = v_o\sin(\theta) = (20.0\ m/s)\sin(60^o) = 17.3\ m/s$.

Thus vertical speeds at later time, $v_y = v_{oy} - gt = v_o\sin(\theta) - gt$

$= (20.0\ m/s)\sin(60^o) - (9.8\ ms^{-2})(3.00\ s) = -12.0\ m/s$.

The minus sign means arrow has attained maximum height before 3.00 seconds, and is already headed back down to the earth.

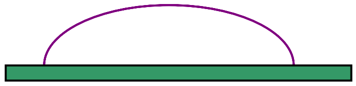
(b) $x = v_{ox}\ t = v_o\cos(\theta)t = (10.0\ m/s)(3.00\ s) = 30.0\ m$

$y = -\frac{1}{2}gt^2 + v_{oy}t + y_o = -4.9t^2 + 17.3t + 0$ so when t = 3.00 seconds,

$y = -4.9\ ms^{-2}(3.00\ s)^2 + 17.3\ m/s(3.00\ s) = 7.8$ meter above the ground.

8. Two youth are playing catch with a ball. Assume that the ball is both tossed and caught 1.25 m above the ground. Ryan tosses the ball with a speed of 5.0 m/s at an initial angle of 45^o. Ignoring air resistance and effects of wind, (a) what is the range of the projectile? (b) Suppose that Ling positions herself precisely at a distance corresponding to the range found in (a). How long does Ling wait to catch the ball?

The assumption means ball starts and ends level.



Then $R = v_{ox} \times$ (twice time to reach maximum height)

$= v_o^2\sin(2\theta) \div g = (5.0\ m/s)^2\sin(90^o)/(9.8\ ms^{-2}) = 2.6$ meter = Range

The time is twice the time to reach maximum height:

$t_{Range} = 2(\ v_{oy}/g\ ) = 2(v_o\sin(\theta)/g) = 0.72$ second.

9. A ball is thrown off a cliff 20.1 m above the ground. Find the time that it takes the ball to reach the ground below if

(a) v_o = 12.5 m/s due east <u>Solution</u>: due east means no y component. Thus projectile looks like our graph 2, free fall in y-direction.

Time to reach ground found by $y = 0 = y_o - \frac{1}{2}gt^2$ where $y_o = 20.0$ m

Thus $t^* = (2y_o\ /g)^{1/2} = (40./9.8)^{1/2} = 2.02 \sim 2.0\ s$.

(b) v_o = 12.5 m/s 18^o <u>south</u> of east. Since velocity already has a negative y component, we predict that the ball will reach the ground a little sooner.

Kinematics: $v_{ox} = (12.5\ m/s)\cos(-18^o) = 11.9\ m/s$

$v_{oy} = (12.5\ m/s)\sin(-18^o) = -3.9\ m/s$

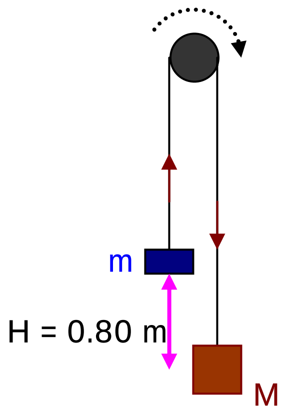
Time to reach ground is again determined by y-equation:

$y = 0 = y_o + v_{oy}t - \frac{1}{2}gt^2 = 20 - 3.9\ t - 4.9t^2$

implies by quadratic formula, $\frac{3.88 \pm \sqrt{(3.88)^2 - 4(20)(-4.9)}}{2(-4.9)}$ = **1.7 s**

(or nonsensical neg. #, -2.4 s, which means that the ball landed 2.4 s before it was thrown.)

the next example combines dynamics and kinematics:



10. Two masses, (16 kg, 4.0 kg) are components of an Atwood machine. Suppose that these masses are held at the same horizontal level, then released. What is the velocity of the coupled masses when the vertical separation between the masses is 80. cm and the masses started from rest?

Let lighter mass be m_1. As it ascends, let up ↑ = + on this side, then from FBD :

$$T - W_1 = m_1 a$$

Let heavier mass be m_2. As it descends, let down ↓ = + on this side, then from FBD :

$$W_2 - T = m_2 a$$

Add the two, eliminating T:

$$W_2 - W_1 = m_1 a + m_2 a = (m_1 + m_2)a \quad \text{Then } a = g\frac{m_1 - m_2}{(m_1 + m_2)} = \frac{12}{20}(9.8) = 5.9 \text{ ms}^{-2}$$

The displacement D equals 40. cm (= ½ H in diagram)—the lighter object ascended 40 cm, the heavier object descended 40 cm—and the masses started from rest, $v_o = 0$, so the handiest formula to use is:

$v^2 - v_o^2 = 2aD$ inserting numbers where we can:

$$v^2 - 0^2 = 2(5.9)(0.4) = 4.72 \text{ (ms}^{-1})^2 \text{ or velocity } v = \sqrt{4.72 \text{ (m/s)}^2} = \underline{2.2 \text{ m/s}}$$