PHYS 111 K
FIRST HOUR EXAM

Solutions

1. In this question, we are asked to consider two vectors :

$$\mathbf{A} = 4\hat{\mathbf{x}} + 6\hat{\mathbf{y}}$$

$$\mathbf{B} = -\hat{\mathbf{x}} - 2\hat{\mathbf{y}}$$

The algebraic sum is simply :

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \left(4\hat{\mathbf{x}} + 6\hat{\mathbf{y}}\right) + \left(-\hat{\mathbf{x}} - 2\hat{\mathbf{y}}\right)$$

Grouping according to unit vector :

$$\mathbf{C} = \left(4\hat{\mathbf{x}} - \hat{\mathbf{x}}\right) + \left(6\hat{\mathbf{y}} - 2\hat{\mathbf{y}}\right) = 3\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$$

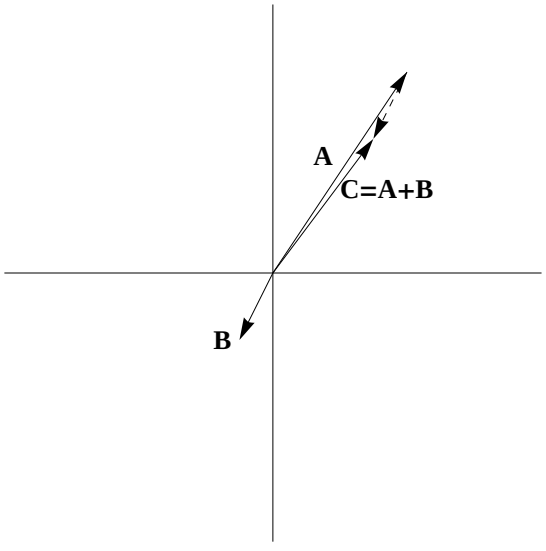
The magnitude of the resultant vector is given by the Pythagorean theorem :

$$| C | = \sqrt{3^2 + 4^2} = 5$$

and the angle the vector C makes with the x axis is :

$$\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

The graphical solution makes use of the "tail to tip" procedure, and should look like :



In this graph, vectors A, B and C are indicated; the dashed line represents vector B moved (without changing its length or orientation in space) such that its tail starts from the tip of vector A.

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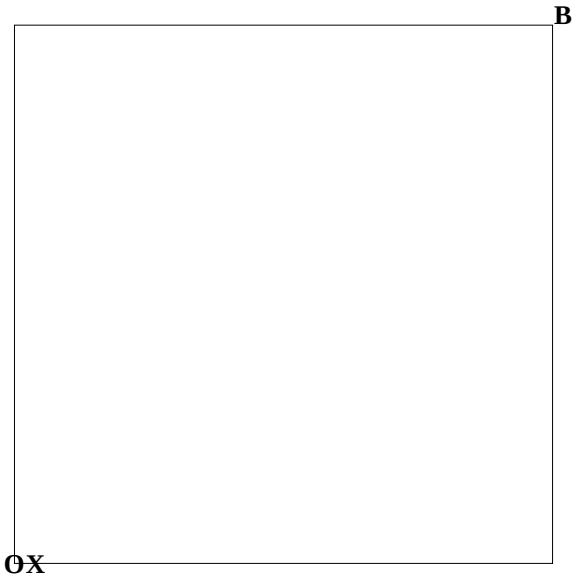
2. This was a problem designed to test your ability to apply the concept of vectors to a scenario. The question asked for the average velocity of the runner. Since velocity is a vector, the average

velocity is the net displacement divided by the time of the trip. Had the runner returned to the starting line (as in the example done in class in the first week), the average velocity would have been zero since there would be no net displacement. In this case, as the figure below shows, the runner started at O and finished at X; the net displacement is 100 m; so the average velocity is 100 m/7 hrs in the positive x direction.



(This diagram is not to scale to allow the point X to be distinguishable from O).

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3. The angle is the same no matter how large we make the radius of the wheel. The angle of the displacement vector is given by :

$$\tan\theta = \left(\frac{y}{x}\right)$$

where y is the vertical displacement of the point and x is the horizontal displacement of the point. The vertical displacement will always be the diameter of the wheel or 2 R (where R is the radius of the wheel); and the horizontal displacement will be the distance a point on the wheel travels in half a revolution, which is half the circumference of the wheel or π R. Thus, no matter the size of the wheel, the angle of the displacement vector with respect to the ground satisfies :

$$\tan\theta = \left(\frac{2\,R}{\pi\,R}\right) = \left(\frac{2}{\pi}\right)$$

which is independent of the wheel radius.

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4. This is a problem in one dimensional motion with uniform acceleration; for this problem, we set the value of acceleration = - 10 m s^{-2}. We are asked to find the time of flight, maximum height achieved, and show a graph of **v** vs. t.

We can find the time of flight in a few ways. First, we can use the equation of motion:

$$y(t) = y_0 + v_0 t + \frac{1}{2} a t^2$$

In this case, the initial y value (the ground) is zero and the initial velocity is + 40 m/s. Once we define the direction of initial velocity as positive, we must define gravity as negative, so for this scenario, our equation of motion becomes :

$$y(t) = v_0 t - \frac{1}{2} g t^2$$

To find the time of flight, we want to know when the rocket will be on the ground (y = 0), so we set y (t) = 0 and find :

$$0 = 40\,t - \frac{1}{2} \cdot 10\,t^2 \Rightarrow t\left(40\,m/s - 5\,m/s^2\,t\right) = 0 \Rightarrow t = 0,\ 8\,s$$

The time of flight is 8 s. We could have also used the fact that the vertical velocity is zero at the maximum time, and found how long it takes the rocket to reach zero velocity :

$$a = \frac{\Delta v}{\Delta t} \Rightarrow t = \frac{v_f - v_0}{a}$$

For this problem, the initial velocity is the launch velocity, and since our analysis ends at the highest point, our final velocity is zero; substituting appropriate numbers gives :

$$t = \frac{0\,m/s - 40\,m/s}{-10\,m/s^2} = 4\,s$$

If 4 s is the time to apex; then 8 s is the total time of flight. We can find the maximum height from the equation of motion, determining the value of y when t = 4 s :

$$y(4\,s) = 0 + 40\,t - \frac{1}{2} g t^2 = 40\,m/s\,(4\,s) - \frac{1}{2} \cdot 10\,m/s^2\,(4\,s)^2 =$$

160 m - 80 m = 80 m

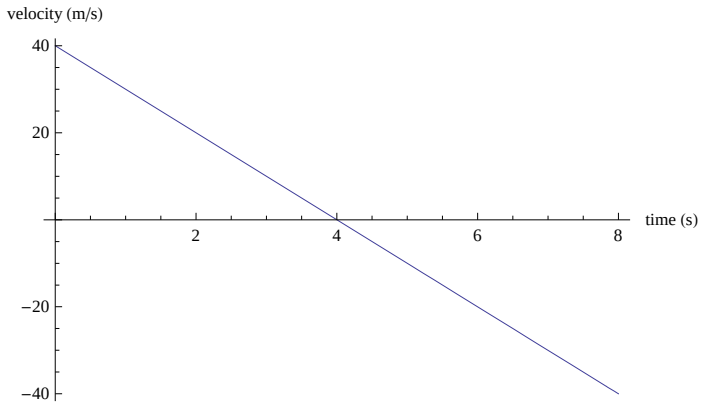
Finally, to plot vertical velocity as a function of time, we begin with the expression :

$$v(t) = v_0 + a t$$

Using the numerical values for this case we have :

$$v(t) = 40\,m/s - 10\,m/s^2\,t$$

The graph of vertical velocity as a function of time is then :



Remember that velocity is a vector; the velocity at lauch is 40 m/s; the velocity at apex is 0, and the velocity at impact is - 40 m/s.The velocity must be negative since the rocket is descending and we have previously set up as the positive direction.

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5. You are asked to use the basic definitions of kinematics to derive two important equations. For purposes of simplicity, let' s set the initial time = 0, therefore we can simply write $\Delta t = t$. We have :

$$v_{av} = \frac{\Delta x}{t} \text{ or } x_f - x_0 = v_{av}\, t \tag{1}$$

We also know in the case of uniform acceleration :

$$v_{av} = \frac{1}{2}(v_f + v_0)\, t$$

Substitute this expression for average velocity into eq. (1) and get :

$$x_f - x_0 = \frac{1}{2}(v_f + v_0)\, t \tag{2}$$

Our definition of acceleration is :

$$a = \frac{\Delta v}{t} = \frac{v_f - v_0}{t} \Rightarrow v_f = v_0 + a\, t$$

Substitute this expression for final velocity into eq. (2) :

$$x_f - x_0 = \frac{1}{2}(v_0 + at + v_0)\, t = \frac{1}{2}(2\, v_0 + a\, t)\, t = v_0\, t + \frac{1}{2} a\, t^2$$

or :

$$x_f = x_0 + v_0\, t + \frac{1}{2} a\, t^2$$

For the second part of this problem, we are asked to eliminate time from the equation of motion to derive :

$$v_f^2 = v_0^2 + 2\, a\, \Delta x$$

We start from the definition of acceleration :

$$a = \frac{\Delta v}{t} \Rightarrow t = \frac{v_f - v_0}{a}$$

Substitute this expression for t into the equation of motion :

$$x_f = x_0 + v_0 \frac{(v_f - v_0)}{a} + \frac{1}{2} a \frac{(v_f - v_0)^2}{a^2}$$

Expanding the squared term and multiplying out the first term on the right :

$$x_f = x_0 + \frac{v_0 v_f - v_0^2}{a} + \frac{1}{2a}\left(v_f^2 - 2 v_f v_0 + v_0^2\right)$$

$$x_f - x_0 = \frac{v_f^2}{2a} - \frac{v_0^2}{2a}$$

$$\Rightarrow 2a(x_f - x_0) = v_f^2 - v_0^2 \Rightarrow v_f^2 = v_0^2 + 2a\Delta x \text{ QED}$$

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6. Starting from the equations of motion, we are asked to derive expressions for time of flight and range. We start with the given equations of motion :

$$y(t) = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$$

$$x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$$

We adopt a coordinate system in which the launch point is at the origin, so both the x and y initial coordinates are zero; defining up as positive, the acceleration in the y direction becomes - g . There is no acceleration in the x direction since we are ignoring friction and no other forces act horizontally. The initial velocity components become :

$$v_{0x} = v_0 \cos\theta;\ v_{0y} = v_0 \sin\theta$$

With these values, our equations of motion simplify to :

$$x(t) = v_0 \cos\theta\, t$$

$$y(t) = v_0 \sin\theta\, t - \frac{1}{2} g t^2$$

We solve for the time of flight by realizing that the motion ends when the projectile hits the ground. In other words, we want to find the time when y (t) = 0. Setting the y equation to zero, we get :

$$0 = v_0 \sin\theta\, t - \frac{1}{2} g t^2 = t\left(v_0 \sin\theta - \frac{1}{2} g t\right)$$

For this equation to hold, we know that t = 0 or

$$t = \frac{2 v_0 \sin\theta}{g}$$

The latter is the time of flight; t = 0 corresponds to the time of launch.

The range is the horizontal distance traveled by the projectile. We determine this by substituting the time of flight into the x equation of motion :

$$\text{Range} = v_0 \cos\theta \left(\frac{2\, v_0 \sin\theta}{g} \right) = \frac{2\, v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\,\theta)}{g}$$

In the last step, we make use of the double angle formula :

$$\sin(2\,\theta) = 2 \sin\theta \cos\theta$$