PHYS 111
HOMEWORK #4--Solutions

Write on only one side of each sheet. To receive full credit for questions involving numerical calculations, use proper units throughout the calculations. Complete solutions and explanations are required for full credit.

We will neglect friction in all questions in this assignment.

1. An object is dropped from rest from a cliff of height H. It is observed that the object completes the last half of the trip in 1 second. Determine the time (in seconds) it takes the object to reach the ground and the height (in meters) of the cliff.

Solution : There are a few ways you could approach this problem. First, we could solve for the time of flight first. The equation of motion for an object dropped from rest from a height H is :

$$y(t) = H - \frac{1}{2} g t^2$$

Then, the time to reach the ground is found by setting y (t) = 0 and obtaining :

$$t = \sqrt{2H/g}$$

So the time it takes for the object to fall the first half of the distance is :

$$t_1 = \sqrt{2(H/2)/g} = \sqrt{H/g}$$

But we also know that the time it takes for the object to fall the entire distance H is simply 1 second more than the time to fall half the height, or :

$$t_2 = \sqrt{2H/g} = t_1 + 1$$

Now, take ratios :

$$\frac{t_1 + 1}{t_1} = \frac{\sqrt{2H/g}}{\sqrt{H/g}}$$

Square both sides :

$$\left(\frac{t_1 + 1}{t_1}\right)^2 = \frac{2H/g}{H/g} = 2$$

Expanding :

$$\frac{t_1^2 + 2t_1 + 1}{t_1^2} = 2 \Rightarrow t_1^2 + 2t_1 + 1 = 2t_1^2$$

Which yields the quadratic equation :

$$t^2 - 2t - 1 = 0$$

(I have dropped the subscript since it is now clear that this is the time for the first half of the trip) :

$$t = \frac{2 \pm \sqrt{4+4}}{2} = 2.4\,\text{s} \tag{1}$$

Since this is the time to fall the first half of the distance, the time to fall the entire height of the cliff is 3.4 s (since it takes 1 s to fall the last half). If we know the time it takes to fall the entire height, then we find the value for the height of the cliff from :

$$H = \frac{1}{2} g t^2 = 0.5 \cdot 9.8 \frac{\text{m}}{\text{s}^2} (3.4\,\text{s})^2 = 57.1\,\text{m}$$

Let' s consider a second way of approaching this problem that will lead us to essentially the same quadratic. We will write the equations of motion for the two halves of motion, being very careful about our coordinate system and about using subscripts properly. Let' s say that the time to complete the first half of the trip is t, and we are told that the time to complete the second half is 1 s. For the first half of the trip, the initial velocity is 0. Also, this is a case where it is convenient to set down as the positive direction. So, when the object has fallen a distance H/2, we can write :

$$H/2 = 0 + 0 + \frac{1}{2} g t^2 \Rightarrow H = g t^2$$

For the second half of the trip, the object has acquired a velocity equal to g t, and this will represent the initial velocity for the second half of the trip. We know the object will travel a distance of H/2 in 1 s, beginning with a velocity of g t. Thus we can write :

$$\frac{H}{2} = (g\,t)(1\,\text{s}) + \frac{1}{2} g (1\,\text{s})^2 \Rightarrow H = 2 g t + g$$

Since these expressions for H are the same, we can equate them and obtain :

$$g t^2 = 2 g t + g \Rightarrow g t^2 - 2 g t - g = 0$$

We can solve for t :

$$t = \frac{2 g \pm \sqrt{4 g^2 + 4 g}}{2 g}$$

If we cancel out common factors of g, we obtain exactly the same quadratic as in eq. (1), which we solve the same way and obtain the same results as above.

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2. An object is launched with initial velocity v_o at an angle θ with respect to a horizontal plane and lands a distance R from the launch point. If the initial velocity is doubled, how far (in terms of R) will it land? If the launch angle is changed to 90 - θ, how far will the object land?

Solution : We begin with the range equation :

$$R = \frac{2\, v_0^2 \sin\theta \cos\theta}{g}$$

If we double the initial velocity, we can write :

$$\frac{R'}{R} = \frac{2\,(2\, v_o)^2 \sin\theta \cos\theta \;/\; g}{2\, v_o^2 \cos\theta \sin\theta / g}$$

Where R' represents the range with the increased velocity. All the terms cancel except for the factor $(2)^2$, so that the new range is 4 times the old range. If we change the angles to 90-θ, we get:

$$R' = \frac{2\, v_o^2 \sin(90-\theta) \cos(90-\theta)}{g}$$

But since sin (90 - θ) = cos θ, and cos (90 - θ) = sin θ, the new range can be written as :

$$R' = \frac{2\, v_o^2 \cos\theta \sin\theta}{g}$$

which is identical to the original expression.

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3. An object is launched on a level plane with initial velocity v_o at an angle θ. What must the angle θ be if the maximum height achieved by the object is equal to its range?

Solution : We start with the two identities established in class for the range (R) and maximum height (H) of a projectile :

$$R = \frac{2\, v_o^2 \sin\theta \cos\theta}{g} \qquad H = \frac{v_o^2 \sin^2\theta}{2\, g}$$

For the case where range equals maximum height, we equate these expressions :

$$\frac{2\, v_o^2 \sin\theta \cos\theta}{g} = \frac{v_o^2 \sin^2\theta}{2\, g}$$

Cancellling common factors of v_o^2, sin θ and g, we obtain:

$$2\cos\theta = \frac{\sin\theta}{2} \Rightarrow 4 = \frac{\sin\theta}{\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}(4) = 76^o$$

We could also use one of the equations I derived in the classnote about how to derive kinematics equations. Equation (9) in that classnote tells us that :

$$\theta = \sin^{-1}\left(\frac{4\,H}{\sqrt{R^2 + 16\,H^2}}\right)$$

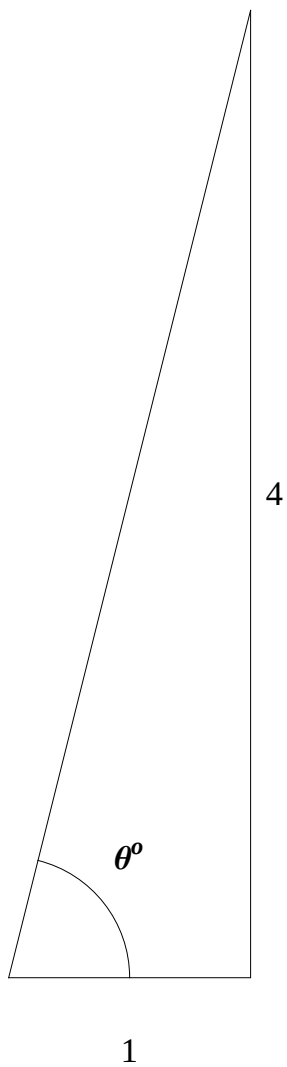
is the launch angle of a projectile with range R and maximum height H. If we set R = H in this equation, we get for our launch angle :

$$\theta = \sin^{-1}\left(\frac{4\,H}{\sqrt{H^2 + 16\,H^2}}\right) = \sin^{-1}\left(\frac{4\,H}{\sqrt{17\,H^2}}\right) = \sin^{-1}\left(\frac{4}{\sqrt{17}}\right) = 76^{o}$$

Finally, we notice that we have the somewhat unexpected result that

$$\tan^{-1}(4) = \sin^{-1}\left(\frac{4}{\sqrt{17}}\right)$$

We can demonstrate this is expected by considering the triangle below :



This diagram, drawn to scale, shows that the tan θ = 4. Sin θ = 4/hypotenuse; you can use the Pythagorean theorem to show that the length of the hypotenuse is $\sqrt{17}$, so that we can see why

$\tan^{-1} 4 = \sin^{-1}\left(4 / \sqrt{17}\right)$.

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4. An object slides off the edge of a horizontal table that is 2 m above a level ground. If the object leaves the edge of the table with a horizontal velocity of 5 m/s, how far from the edge of the table will the object land on the ground?

Solution : In this problem, we know there are no forces in the x direction, so that the horizontal component of motion of the object will be constant. Thus, if we know the time of flight, we know the total horizontal distance traveled will be v_x t.

There is a force in the y direction, gravity, so the motion in the y direction will be accelerated. We begin by solving for the time of flight by writing the equation of motion in the y direction :

$$y(t) = \frac{1}{2} g t^2$$

In this situation, we set down as the positive axis, so that the ground is y = 2 m. Solving for the time of flight :

$$2\,\text{m} = \frac{1}{2} g t^2 \Rightarrow t = \sqrt{4/g} = 0.64\,\text{s}$$

Traveling at a constant horizontal velocity of 5 m/s, an object will travel :

$$\text{horizontal distance} = 5\,\text{m/s} \cdot 0.64\,\text{s} = 3.2\,\text{m}$$

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5. Consider the same situation as in problem 4, except now we do not have numbers. Call the height of the table H and the speed of the object as it slides off the edge v. Derive expressions for a) the time of flight, and b) the distance from the edge of the table where the object strikes.

Solution : The time for an object to fall from rest through a height H is :

$$H = \frac{1}{2} g t^2 \Rightarrow t = \sqrt{2H/g}$$

If the initial horizontal velocity is vx, then the total distance traveled is :

$$\text{Distance} = v_x t = v_x \sqrt{2H/g}$$

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6. An object is launched from the edge of a cliff that is 100 m above the ground. The object has an initial velocity of 30 m/s that is directed at an angle 40 degrees with respect to the edge of the cliff. (See the figure for problem 22 on p. 95; this figure represents the general idea, except the numbers we are using here are very different). Determine :

a) The time of flight

b) The maximum height the object achieves above the ground.

c) The horizontal distance the object lands from the base of the cliff.

d) The x and y components of the velocity at the instant just before impact.

e) The magnitude of the velocity and the angle the velocity vector makes with the ground.

Five points for each part.

Solutions : We adopt a coordinate system where up is positive, the ground is at y = 0 (so that the top of the cliff is at y = H). We write the equations of motions in the x and y directions :

$$x(t) = v_{0x}\, t = v_o \cos\theta\, t$$

$$y(t) = H + v_{oy}\, t - \frac{1}{2} g\, t^2 = H + v_o \sin\theta\, t - \frac{1}{2} g\, t^2$$

For this problem we have :

$H = 100\,\text{m}$

$v_o = 30\,\text{m/s}$

$\theta = 40^o$

So we have :

$x(t) = 30 \cos 40\, t = 23\, t$

$y(t) = 100 + 30 \sin 40\, t - 4.9\, t^2 = 100 + 19.3\, t - 4.9\, t^2$

We find the time of flight by setting y (t) = 0 and solving the quadratic :

$$4.9\, t^2 - 19.3\, t - 100 = 0$$

This has the solution :

$$t = \frac{19.3 \pm \sqrt{19.3^3 + 4 \cdot 100 \cdot 4.9}}{9.8} = 6.9\,\text{s}$$

(the other solution yields a negative result which is not physically meaningful).

b) The maximum height of the object above its launch point is :

$$H = \frac{(v_o \sin\theta)^2}{2\,g} = \frac{\left(30\,\frac{\text{m}}{\text{s}} \sin 40\right)^2}{2 \cdot 9.8\,\frac{\text{m}}{\text{s}^2}} = 19\,\text{m}$$

The projectile rises 19 m above the cliff, or will be 119 m above the ground below the cliff.

c) Since there are no forces in the x direction, the horizontal component of motion is constant, so that we have :

$$\text{horizontal distance} = v_o \cos\theta \cdot \text{time of flight} = 30 \cos 40 \cdot 6.9\,\text{s} = 159\,\text{m}$$

d) The x component of motion is constant, so is 23 m/s as computed above. The y component of velocity can be determined from :

$$v_y(t) = v_{oy} + at = 30 \sin 40 - 9.8\,\frac{\text{m}}{\text{s}^2} \cdot 6.9\,\text{s} = -48.3\,\text{m/s}$$

e) The total velocity is $\sqrt{v_x^2 + v_y^2} = \sqrt{(23\,m/s)^2 + (-48.3\,m/s)^2}$ = 53.5m/s

The direction of the velocity vector is found from :

$$\tan\theta = \frac{v_y}{v_x} = -\frac{48.3\,\text{m/s}}{23\,\text{m/s}} \Rightarrow \theta = 64.5^{o} \text{ as measured clockwise upward from the negative x axis.}$$

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7. As we have discussed in class, one of the most important elements of physics is learning how to combine equations to derive new expressions. This is often useful when we can measure certain variables, but not others. Now, consider the situation described in problem 6 (except we don' t have numbers.) The height of the cliff is H.An object is launched at an angle θ with respect to the edge of the cliff. If the object lands a distance D from the base of the cliff, show that its maximum height above the ground is given by :

$$H_{max} = H + \frac{D^2 \tan^2\theta}{4\,(H + D\tan\theta)}$$

Note that the expression does not involve velocity. This means you will have to use the relationships for projectile motion that we derived in class, and substitute appropriately for velocity.

Solution : We begin by writing the equations of motion for this projectile:

$$y(t) = H + v_o \sin\theta\, t - \frac{1}{2} g\, t^2$$

$$x(t) = v_o \cos\theta\, t$$

As we did in class, we can eliminate t from these equations and find that :

$$t = \frac{x}{v_o \cos\theta}$$

Substituting this expression for t into the y (t) equation allows us to find y (x) :

$$y(x) = H + x\tan\theta - \frac{g\,x^2}{2\,v_o^2 \cos^2\theta}$$

Now, we know that when y = 0, x = D so we can obtain :

$$0 = H + D\tan\theta - \frac{g\,D^2}{2\,v_o^2 \cos^2\theta} \tag{2}$$

This is almost what we are looking for, but we need to eliminate v_o from the equation. We can do this by recalling that the height above the launch point, which we will call H' in this case, is given by:

$$H' = \frac{v_o^2 \sin^2\theta}{2\,g} \Rightarrow v_o^2 = \frac{2\,g\,H'}{\sin^2\theta}$$

Now, we substitute this expression for H' into equation (2) and obtain:

$$0 = H + D\tan\theta - \frac{g\,D^2}{2\left(\frac{2\,g\,H'}{\sin^2\theta}\right)\cos^2\theta} = H + D\tan\theta - \frac{D^2\tan^2\theta}{4\;H'} \tag{3}$$

in the last step, we cancel common factors of g, and recall that $\tan\theta = \sin\theta/\cos\theta$. Now, we solve equation (3) in terms of H' :

$$\frac{D^2\tan^2\theta}{4\;H'} = H + D\tan\theta$$

Multiply both sides by H' and divide through by $H + D\tan\theta$:

$$H' = \frac{D^2\tan^2\theta}{4\,(H + D\tan\theta)}$$

Finally, remember that H' is the height above the launch point, so that the height above the ground is just H + H', or :

$$H_{max} = H + \frac{D^2\tan^2\theta}{4\,(H + D\tan\theta)}$$