PHYS 111
HOMEWORK #3-- Solutions

1. For each of these functions, we wish to compute :

$$\text{slope} = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{\Delta x}$$

for different values of Δx. For ease of computation, I will use 2.0 in each case for the value of x_1 .

a) for $f(x) = x^2$:

$$\text{slope} = \frac{(2.1)^2 - (2.0)^2}{0.1} = \frac{4.41 - 4.00}{0.1} = 4.1$$

$$\text{slope} = \frac{(2.01)^2 - (2.00)^2}{0.01} = \frac{4.04 - 4.00}{0.1} = 4.01$$

$$\text{slope} = \frac{(2.001)^2 - (2.000)^2}{0.001} = \frac{4.004 - 4.000}{0.001} = 4.001$$

Clearly, as Δx get smaller, we expect the slope to tend toward 4, which is the expected value of the derivative of x^2 evaluated at 2, since :

$$\frac{d}{dx} x^2 = 2x$$

b) for $f(x) = x^3$:

$$\text{slope} = \frac{2.1^3 - 2.0^3}{0.1} = \frac{9.26 - 8.00}{0.1} = 12.61$$

$$\text{slope} = \frac{2.01^3 - 2^3}{0.01} = \frac{8.121 - 8.00}{0.01} = 12.1$$

$$\text{slope} = \frac{2.001^3 - 2.00^3}{0.001} = \frac{8.012 - 8}{0.001} = 12.001$$

The slope is trending toward 12 as Δx becomes smaller; the derivative of x^3 is :

$$\frac{d}{dx} x^3 = 3x^2$$

which equals 12 when x = 2.

c) for $f(x) = x^4$:

Following a similar procedure :

$$\text{slope} = \frac{2.1^4 - 2.0^4}{0.1} = \frac{19.448 - 16.000}{0.1} = 34.48$$

$$\text{slope} = \frac{2.01^4 - 2.00^4}{0.01} = \frac{16.322 - 16.000}{0.01} = 32.24$$

$$\text{slope} = \frac{2.001^4 - 2.000^4}{0.001} = \frac{16.0320 - 16.0000}{0.001} = 32.024$$

The slope tends toward a values of 32 as Δx becomes smaller; this is expected since:

$$\frac{d}{dx} x^4 = 4 x^3$$

which equals 32 when x=2.

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2. The problem I wanted you to do was #77 on p. 68 (I transposed the question and page numbers). We are asked to find the constant acceleration of the rocket given information on the time when the bolt falls off (t = 4 s) and how long it takes the bolt to hit the ground (another 6 s). We are interested in writing the equation of motion of the bolt :

$$y_{bolt}(t) = y_o + v_o t - \frac{1}{2} g t^2$$

While this is the standard equation of motion, we need to be careful how we define these parameters, in other words, what values will we use for y_o, v_o, and t? Let's say the equation of motion above starts at the moment when the bolt falls off; this means we have to figure out how high the rocket has travelled in the first 4s of motion, and the speed the rocket has acquired at that time.

To do this, we consider the first phase of motion, the first 4s when the bolt is still on the rocket. In this time, the rocket will rise a distance:

$$y_{rocket}(t) = y_o + v_o t + \frac{1}{2} a t^2$$

In this phase, the initial starting position is on the ground (so y_o = 0, and the rocket starts from rest, so v_o = 0); thus, in the first 4s of motion, the rocket will reach a height of 1/2 a $(4\,s)^2$ or 8 a meters; this will be the value of y_o in the y_{bolt} equation of motion.

Now, what's the velocity of the rocket at t= 4s? Since the initial velocity was zero for this phase, the velocity at the end of 4s is simply 4 a m/s, and this will be the value of v_0 to use in the y_{bolt} equation. Substituting these values, we get:

$$y_{bolt}(t) = 8\,a + 4\,a\,t - \frac{1}{2} g t^2$$

(once the bolt leaves the rocket, the only force it experiences is gravity, so its acceleration is - g). Now, we know that the bolt hits the ground 6 s after it leaves the rocket. We set y_{bolt}(t) = 0 and set t = 6s to obtain:

$$0 = 8\,a + 4\,a\,(6\,s) - \frac{1}{2} g\,(6\,s)^2$$

This is now a single variable equation in a; we solve for a :

$$8\,a + 24\,a = 18\,g \Rightarrow 32\,a = 18\,g \Rightarrow a = 9\,g/16 \text{ or } 5.5\,\text{m}/\text{s}^2$$

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3. We are given the model equation :

$$v = a\left(1 - e^{-b\,t}\right)$$

where v is speed in m/s, t is time in s, and a and b are constants determined to be :

$$a = 11.81\ \text{m/s} \qquad b = 0.6887\ \text{s}^{-1}$$

(make sure you do the proper dimensional analysis to convince yourself these are the appropriate units for these constants.)

We are asked to find the acceleration at three different times; if you know how to differentiate exponential functions, you get :

$$\text{accel} = \frac{dv}{dt} = a\,b\,e^{-b\,t}$$

Evaluated at t = 1 s yields :

$$\text{accel} = (11.81\ \text{m/s}) * (0.6887/\text{s})\, e^{-0.6887/\text{s} * 1\ \text{s}} = 4.08\ \text{m}\big/\text{s}^2$$

We evaluate at t = 3 s and 5 s by substituting these values of t into the accel equation above, and obtain :

$\text{accel} = 1.03\ \text{m}\big/\text{s}^2$ at $t = 3$ s and $\text{accel} = 0.26\ \text{m}\big/\text{s}^2$ at $t = 5$ s,

showing that the runner is approaching a constant speed as the race evolves.

(Note : sort of a bad choice for the author to give the constants as "a" and "b" when we are solving an equation for "a...ccel")

If you don' t know how to differentiate these types of functions, we can determine the acceleration by taking the slope of the tangent line in the speed vs. time graph, or by computing :

$$\text{acceleration} = \text{slope} = \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

Since the race is only about 10 s in length, let' s choose a value of Δt = 0.01 s; we' ll see how these results compare to the values determined via calculus methods above (and that will help determine whether our choice for Δt was small enough). For t = 1 s we have :

$$\text{acceleration} = \frac{a\left(1 - e^{-b\,(1.01\ \text{s})}\right) - a\left(1 - e^{-b\,(1.0\ \text{s})}\right)}{0.01\ \text{s}} = \frac{a\left(e^{-b\,(1.0\ \text{s})} - e^{-b\,(1.01\ \text{s})}\right)}{0.01\ \text{s}} = 4.07\ \text{m}\big/\text{s}^2$$

For t = 3 s :

$$\text{acceleration} = \frac{a\left(e^{-(3.0\ \text{s})\,b} - e^{-(3.01\ \text{s})\,b}\right)}{0.01\ \text{s}} = 1.03\ \text{m}\big/\text{s}^2$$

For t = 5 s :

$$\text{acceleration} = \frac{a\left(e^{-(5.0\ \text{s})\,b} - e^{-(5.01\ \text{s})\,b}\right)}{0.01\ \text{s}} = 0.26\ \text{m}\big/\text{s}^2$$

We have good agreement between the two approaches suggesting that using even smaller values of Δt will yield even better agreement.

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4. We will be working with the equation :

$$v^2 = \frac{2\,P}{m}\,t$$

where v is the speed (measured in m/s), P is the power (measured in the MKS unit of watts), m is the mass (measured in kg) and t is the time (measured in s). We are told that for this case, P has a value of $3.6 \cdot 10^4$ W and m = 1200 kg. We are asked:

a) Compute v when t = 10 s and t = 2 s :

We simply substitute these values of t into the equation above :

$$v = \sqrt{2 \cdot 3.6 \times 10^4 \, \mathrm{W} \cdot 10 \, \mathrm{s} / 1200 \, \mathrm{kg}} = \sqrt{600 \, \mathrm{m}^2 / \mathrm{s}^2} = 24.5 \, \mathrm{m/s}$$

For part b), you could substitute t = 20 s, or just note that our answer to this part must be $\sqrt{2}$ larger or a speed of 34.6 m/s.

b) and c) We write the speed as :

$$v = (2\,P/m)^{1/2}\, t^{1/2}$$

The acceleration is just the time derivative of v; and this expression of v is written in the familiar format of v = c t^n where in this case the constant is $\sqrt{2\,P/m}$ and n = 1/2. Therefore, the acceleration of this particle is:

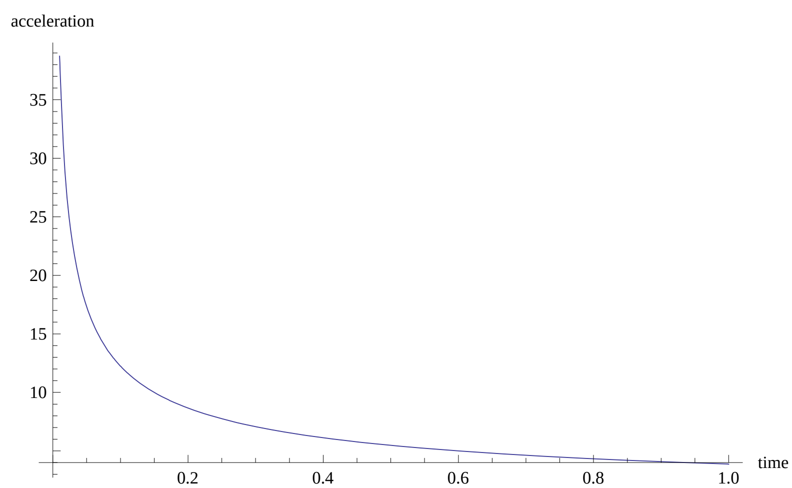
$$a = \frac{dv}{dt} = \frac{1}{2}(2\,P/m)^{1/2}\, t^{-1/2} = \frac{\sqrt{2\,P/\,m\,t}}{2} = \sqrt{P/2\,m\,t}$$

Evaluating this expression at t = 1 s and t = 10 s yields :

$$a = \sqrt{3.6 \cdot 10^4 \, \mathrm{W} / (2 \cdot 1200 \, \mathrm{kg} \cdot 1 \, \mathrm{s})} = 3.88 \, \mathrm{m} / \mathrm{s}^2$$

The acceleration at t = 10 s will be $\sqrt{10}$ smaller than at t=1s, or a(t=10s) = 1.22 m/s^2

d) The key here is that the acceleration term has time in the denominator; this means that as t → 0 (right after motion begins), this models predicts almost infinitely large, a result which is not physically meaningful. Below is a graph of the acceleration vs. time according to this model :



According to this model, a passenger in the vehicle would feel a tremendous change of acceleration in the first few tenths of a second of motion; this is not likely realistic, suggesting that the model is not valid for the first few tenths of a second of motion. Some of you know that the time rate of

change of acceleration is called jerk, and is defined as
j = da/dt.

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5. We can write the person' s motion in vector notation as :

$$\mathbf{R} = 3.1\,\hat{\mathbf{y}} - 2.4\,\hat{\mathbf{x}} - 5.2\,\hat{\mathbf{y}}$$

where we use a standard Cartesian coordinate system in which we denote east and north as postiive, west and south as negative and R is the resultant vector. Summing components we obtain simply :

$$\mathbf{R} = -2.4\,\hat{\mathbf{x}} - 2.1\,\hat{\mathbf{y}}$$

The magnitude of the resultant is given by the Pythagorean theorem::

$$|\mathbf{R}| = \sqrt{(-2.4\text{ km})^2 + (-2.1\text{ km})^2} = 3.2\text{ km}$$

The ending point lies in the third quadrant we can describe the direction of the resultant vector using:

$$\tan\theta = \frac{|r_y|}{|r_x|} = \frac{2.1\text{ km}}{2.4\text{ km}} \Rightarrow \theta = 41^{o}\text{ south of west}$$

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6. The point P moves both in the x and y directions. Its y displacement is simply the diameter of the wheel, 90 cm. It x displacement is the total distance a point on the wheel will travel in one - half revolution, or a distance equal to π R (where R is the radius of the wheel). (Remember that the circumference of a circle is 2 π R, so this is the distance traveled by a point on the wheel in one full revolution; thus π R is the distance traveled in half a revolution.)

Thus, the magnitude of the displacement vector is :

$$|\mathbf{R}| = \sqrt{(45\,\pi\text{ cm})^2 + (90\text{ cm})^2} = 168\text{ cm}$$

The angle made by the displacement vector with respect to the ground is:

$$\tan\theta = \frac{90\text{ cm}}{45\,\pi\text{ cm}} = 0.64 \Rightarrow \theta = 32.5^{o}$$