PHYS 111 K
HOMEWORK #4

Solutions

1. We need to establish a three dimensional coordinate system for this problem. Let' s call the plane on which the man walks to be the z = 0 plane (the standard Cartesian x - y plane). We could have just as easily set the ground (where the coin hits) as z = 0. Then, in unit vector notation, the final position of the coin is simply :

$$\mathbf{R} = 1000\,\hat{\mathbf{x}} + 2000\,\hat{\mathbf{y}} - 100\,\hat{\mathbf{z}}$$

The magnitude of this displacement is given by the Pythagorean theorem :

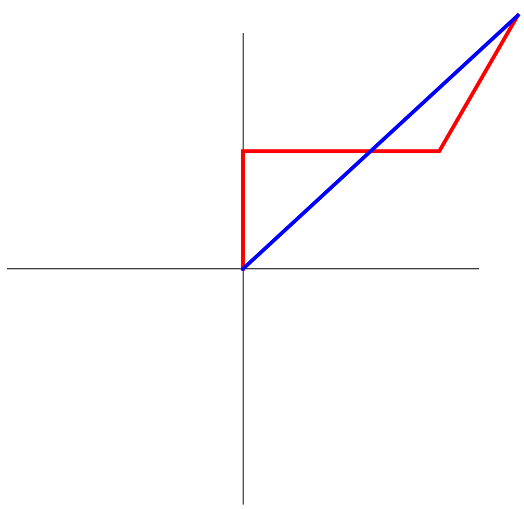
$$|\,\mathbf{R}\,| = \sqrt{(1000\text{ m})^2 + (2000\text{ m})^2 + (-100\text{ m})^2} = 2238\text{ m}$$

If the man returns to his starting position, his net displacement is zero.

2. As above, we use the 3 dimensional version of the Pythagorean theorem to determine the magnitude of the displacement vector :

$$|\,\mathbf{R}\,| = \sqrt{(8\text{ m})^2 + (10\text{ m})^2 + (12\text{ m})^2} = 17.5\text{ m}$$

3. First, let' s graph the motion :



The red lines show each of the trip segments, the blue line shows the resultant vector. It is clear how we should write the first two segments in unit vector notation (30 y and 50 x). We have to determine the x and y components of the final segment. If the final segment is 30 degrees east of north (which is equivalent to 60 degrees north of east) , then we can use trig to write :

x component of final segment = 40 cos 60^o = 20 km;

y component = 40 sin 60^o = 34.6 km.

Therefore, the resultant can be written :

$$\mathbf{R} = 30\,\hat{\mathbf{y}} + 50\,\hat{\mathbf{x}} + \left(20\,\hat{\mathbf{x}} + 34.6\,\hat{\mathbf{y}}\right) = 70\,\hat{\mathbf{x}} + 64.6\,\hat{\mathbf{y}}$$

The magnitude of this vector is :

$$|\,\mathbf{R}\,| = \sqrt{(70\text{ km})^2 + (64.6\text{ km})^2} = 95.2\text{ km}$$

The angle of this vector with respect to the + x axis is :

$$\tan\theta = \frac{64.6\text{ km}}{70\text{ km}} \Rightarrow \theta = \tan^{-1}\left(\frac{64.6}{70}\right) = 42.7^{\text{o}}$$

4. All of these can be differentiated according to the exponent rule :

$$\frac{d}{dx}(c\,x^n) = n\,c\,x^{n-1}$$

where c is a constant. Therefore,

a) $f(t) = t + \frac{1}{t} \Rightarrow f'(t) = 1 - \frac{1}{t^2}$; $\left(\text{note}: \frac{1}{t} = t^{-1}\right)$

b) $f(x) = x^2 - 2\,x^{-3} + \frac{4}{x^4} \Rightarrow f'(x) = 2\,x + 6\,x^{-4} - 16\,x^{-5}$

c) $f(s) = a\,s^2 + c\,s + d \Rightarrow f'(s) = 2\,a\,s + c$ (here, s is the variable)

d) $f(x) = \left(2\,x^2 + 3\,x + 1\right)^3$

We can use the chain rule by setting u = $2\,x^2 + 3\,x + 1$ and:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

So, if f (u) = u^3, $\frac{df}{du} = 3\,u^2$ and we have:

$$\frac{df}{dx} = 3\,u^2 \cdot \frac{du}{dx} = 3\,u^2\,(4\,x + 3) = 3\left(2\,x^2 + 3\,x + 1\right)^2 (4\,x + 3)$$

If you do not know the chain rule, expand the function by directly computing its cube :

$$\left(2\,x^2 + 3\,x + 1\right)^3 = 8\,x^6 + 36\,x^5 + 66\,x^4 + 63\,x^3 + 33\,x^2 + 9\,x + 1$$

so that f' (x) becomes :

$$f'(x) = 48\,x^5 + 180\,x^4 + 264\,x^3 + 189\,x^2 + 66\,x + 9$$

e) $f(t) = \frac{1}{1+t} \Rightarrow f'(t) = \frac{-1}{(1+t)^2}$

5. This is another problem that we divide into two segments (each of the two halves of the trip). This problem provides us with the opportunity to choose carefully how we define the initial values of each portion of the trip.

For the first portion (falling the first half of the distance), we can set the initial value of y and the initial value of t = 0. The initial vertical velocity here is zero, so the distance the object travels is one half the total height (let' s call this H/2), and let' s call the time to reach the half way point t_1, then we can use the y equation of motion to show easily that we can express this time as:

$$t_1 = \sqrt{\frac{2\,(H/2)}{g}} = \sqrt{\frac{H}{g}}$$

To find the velocity at t_1, recall that the velocity at the beginning of this segment was zero, so when the object reaches the half way point, its velocity has increased to g t_1.

Now, we are ready to begin our analysis of the second half of the trip. For the second half of the trip, we can reset our coordinates such that the initial position is 0 and the final position (bottom of the cliff) is H/2. Thus, down is positive, so the initial velocity at this new starting point is:

$$v\,\text{initial for the second half of the trip} = g\,t_1 = g\sqrt{H/g} = \sqrt{H\,g}$$

We can write the y equation of motion for this second half of the trip (remembering that this phase takes only 1 s) as :

$$\frac{H}{2} = \sqrt{H\,g}\;(1\,s) + \frac{1}{2}\,g\,(1\,s)^2$$

(Remember, we defined down to be positive, so both velocity and the acceleration due to gravity have positive signs in this coordinate system). This equation becomes :

$$\frac{H}{2} = \sqrt{H\,g} + \frac{1}{2}\,g$$

or :

$$\sqrt{H\,g} = \frac{H}{2} - \frac{g}{2}$$

Square both sides :

$$H\,g = \frac{H^2}{4} - \frac{g\,H}{2} + \frac{g^2}{4} \Rightarrow \frac{H^2}{4} - \frac{3\,g\,H}{2} + \frac{g^2}{4} = 0$$

This is a quadratic equation in H which has the solutions :

$$H = \frac{\frac{3\,g}{2} \pm \sqrt{\frac{9\,g}{4}^2 - 4\left(\frac{1}{4}\cdot\frac{g^2}{4}\right)}}{1/2} = \frac{\frac{3\,g}{2} \pm \sqrt{2\,g^2}}{1/2} = 5.83\,g \text{ or } 0.17\,g$$

Thus, the height of the cliff is 5.83 (9.81) m = 57.2 m. It is easy to show that the second solution, H = 1.67 m is not physically realistic since an object would fall through this distance in 0.58 s (and the second half of the trip took 1 s).

We find the time of flight by computing the time it takes to fall from rest through a distance of 57 .2

m:

$$t = \sqrt{2\,H/g} = \sqrt{2 \cdot 57.2\,\text{m}/9.81\,\text{m}/\text{s}^2} = 3.41\,\text{s}$$

Let' s check these data. If this value for the total time is correct, then the object should have taken 2.41 s to fall half the total distance. In 2.41 s, an object starting at rest will fall a distance :

$$s = \frac{1}{2}\,g\,t^2 = \frac{1}{2} \cdot 9.81\,\text{m}/\text{s}^2\,(2.41\,\text{s})^2 = 28.6\,\text{m}$$

and this is consistent with the statement of the problem.

Now, let’s analyze the second half of the trip. At the end of the first phase (at t = 2.41 s), the object' s speed will be

$$v(2.41\text{s}) = 2.41\text{ s} \cdot 9.81\, m/s^2 = 23.6\text{ m/s}$$

Now, how far will an object fall in one second if it starts with an initial velocity of 23.6 m/s?

$$s = v_0\,t + \frac{1}{2}\,g\,t^2 = 23.6\,\text{m}/\text{s}\,(1\,\text{s}) + \frac{1}{2} \cdot 9.8\,\text{m}/\text{s}^2\left(1\,\text{s}^2\right) = 28.6\,\text{m}$$

Adding the two halves of the trip gives us the expected figure of 57.2 m, so we can conclude we have correctly found the time of travel and height of the cliff.

6. This problem models a very important skill in physics : proportional reasoning. We can start with the range equation :

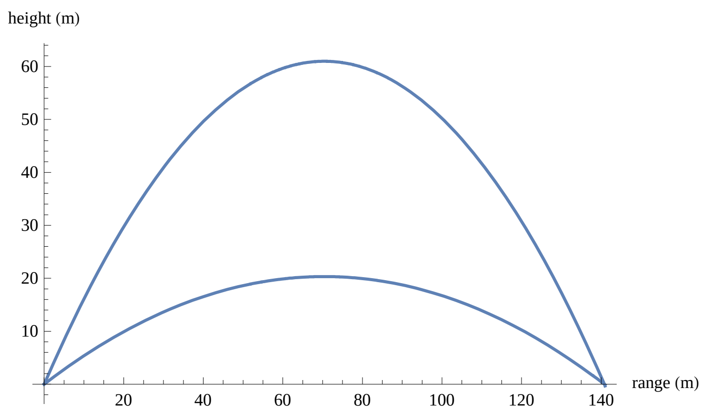
$$\text{Range} = \frac{2\,v_0^2 \cos\theta \sin\theta}{g}$$

First, we are asked to determine the range if the initial velocity is doubled. We do not need to know a launch angle, nor do we need to make up values for the angle and initial velocity to solve this problem numerically. Notice that range depends on the square of the initial velocity, so that if this value is doubled, then the range will increase by 2^2 , so the range would increase by a factor of 4.

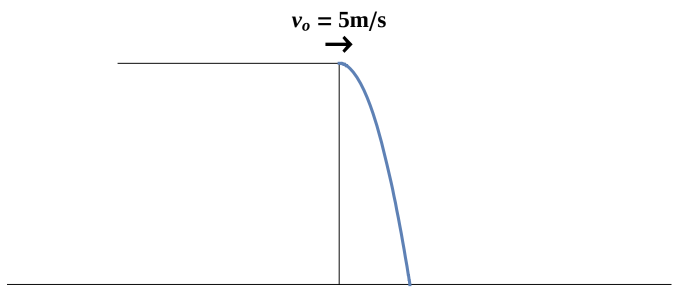
Now, how would the range change if the angle becomes 90-θ? Substitute this new value of angle into the range equation:

$$\text{Range} = \frac{v_0^2 \cos(90-\theta) \sin(90-\theta)}{g}$$

But, you should remember from trig that sin (90 - θ) = cos θ and also that cos (90 - θ) = sin θ. Making these substitutions returns us to our original range equation, and the distance traveled is unaffected by this change in angle. The figure below shows the trajectories of two projectiles launched with the same initial velocity; one has a launch angle of 60, the other 30 :



7. Let' s begin with a diagram depicting the situation :



The object slides off the table with an initial horizontal velocity of 5 m/s. Since there are no forces acting in the x direction, we expect that the x component of velocity will not change, and it will travel at a rate of 5 m/s until it hits the ground.

The initial y velocity is zero. Once the object leaves the table, it will begin accelerating downward, so we can determine its time of flight by using the y equation of motion. Let' s make down the positive direction, therefore its initial y value is zero, its initial vertical velocity is zero, and the direction of gravity is positive. Then, the y equation becomes :

$$y(t) = y_0 + v_{0y}\, t + \frac{1}{2} g t^2 = \frac{1}{2} g t^2$$

The time for the object to fall two meters is :

$$t = \sqrt{2H/g} = \sqrt{2 \cdot 2\,m/9.8\,m/s^2} = 0.64\,s$$

If the object travels at 5 m/s for this time, its horizontal distance is 0.64 m/s 5 s = 3.2 m.

8. Now we will solve the problem without numbers. The x equation of motion tells us :

$$x = v_{0x}\, t$$

The y equation of motion gives us the time of flight :

$$t = \sqrt{2H/g}$$

so the range is simply :

$$\text{Range} = v_{ox} \sqrt{2H/g}$$

9. This is another problem in which we use the equations of motion. In this case, it makes sense to set up as the positive direction, therefore the initial conditions become :

$$y_0 = 100\,\text{m};\ v_{0\,x} = 30\cos 40;\ v_{0\,y} = 30\sin 40;\ g = -9.81\,\text{m/s/s}$$

Writing the equations of motion :

$$x(t) = v_0 \cos\theta\, t$$

$$y(t) = y_0 + v_0 \sin\theta\, t - \frac{1}{2} g t^2$$

a) We find the time of flight by setting y = 0 and get :

$$0 = y_0 + v_0 \sin\theta\, t - \frac{1}{2} g t^2$$

This is a quadratic equation in t with solutions :

$$t = \frac{v_0 \sin\theta \pm \sqrt{v_0^2 \sin^2\theta + 2\,g\,H}}{g}$$

(How does this equation for time of flight compare to the expression derived for projectile motion on a level plane? Can you identify the differences between the two equations? Will this equation reduce to the expression on a level plane if you set H = 0?) If we look at the discriminant (the stuff inside the square root), we can see that the magnitude of the discriminant is great than v sin θ. Therefore, we know that we should take the positive branch to get the physically realistic solution (the negative branch will yield a negative time). Thus, the time of flight is :

$$t = \frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2\,g\,H}}{g}$$

Now, AND ONLY NOW do you insert numbers :

$$t = \frac{30\sin 40\,\text{m/s} + \sqrt{(30\sin 40\,\text{m/s})^2 + 2\left(9.81\,\text{m}/\text{s}^2\right)(100\,\text{m})}}{9.81\,\text{m}/\text{s}^2}$$

$$t = \frac{19.3\,\text{m/s} + \sqrt{2334\,\text{m}^2/\text{s}^2}}{9.8\,\text{m}/\text{s}^2} = 6.9\,\text{s}$$

b) We know that at maximum altitude the vertical velocity of the projectile is zero. We can find the height above the cliff using :

$$v_{fy}^2 = v_{0\,y}^2 + 2\,a\,y$$

Where we are interested in the final and initial velocities in the vertical direction (horizontal motion will not contribute to the height). The acceleration in this case is - g, and we are interested in computing y, which will represent the highest point above the cliff. Since the final vertical velocity (for this portion of the trip) is zero, we have :

$$0 = v_{0y}^2 - 2gy \Rightarrow y = \frac{v_{0y}^2}{2g} = \frac{(30 \sin 40\,\text{m/s})^2}{2 \cdot 9.8\,\text{m}/\text{s}^2} = 19\,\text{m}$$

Since this is the height above the cliff, the maximum height above the ground is 119 m.

c) To find the horizontal distance traveled, remember that the horizontal component of velocity does not change (since there are no forces acting in the horizontal direction). Therefore, the horizontal distance is computed simply by multiplying the constant horizontal velocity by the time of flight, so :

$$\text{Range} = v_{0x}\,t = 30 \cos 40\,t = 23\,\text{m/s} \cdot 6.9\,\text{s} = 159\,\text{m}$$

d) We have just computed the x component of velocity at impact (since it never varies throughout the trajectory). The vertical component of velocity does change because the force of gravity does act in the vertical direction. Starting from the definition of acceleration :

$$a = \frac{\Delta v}{t} \Rightarrow v_{fy} = v_{0y} + a\,t = v_{0y} - g\,t$$

The y velocity at impact is then :

$$v_{fy} = 30 \sin 40\,\text{m/s} - 9.8\,\text{m}/\text{s}^2 \cdot 6.9\,\text{s} = -48.3\,\text{m/s}$$

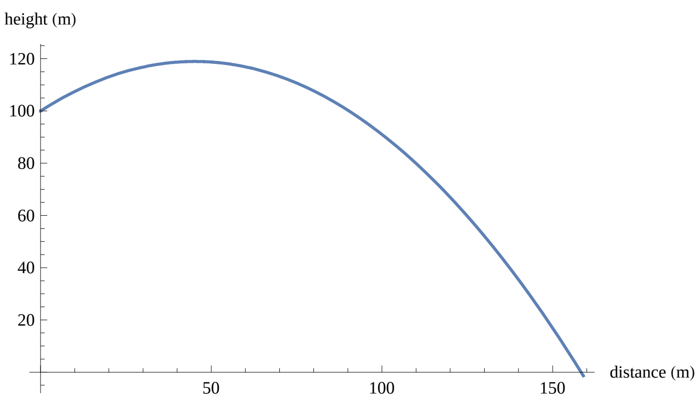
e) The magnitude of the final velocity vector is :

$$|\mathbf{v}| = \sqrt{(23\,\text{m/s})^2 + (-48.3\,\text{m/s})^2} = 53.5\,\text{m/s}$$

The angle the velocity vector makes with the horizontal is :

$$\tan\theta = \frac{|v_y|}{v_x} = \frac{48.3\,\text{m/s}}{23\,\text{m/s}} \Rightarrow \theta = 64.5^o \text{ clockwise up from the x axis.}$$

Plotting the trajectory :



The angle between the x axis and the curve in the graph is 64.5^o just at the moment of impact.