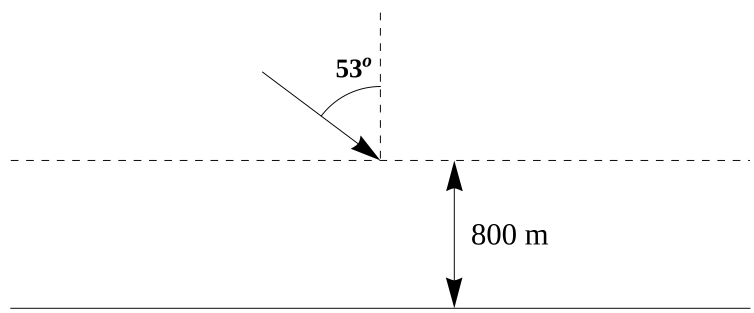
PHYS 111 K
HOMEWORK #5-- SOLUTIONS

1. The initial velocity of the released bomb is equal to the plane' s speed at the moment of release; the components of the initial velocity are :

$$v_{ox} = v_o \sin 53^o \qquad v_{oy} = -v_o \cos 53^o$$

where v_o is the speed of the plane, and the y initial velocity is negative since the bomber is moving down. We wil adopt a coordinate system where up is positive, so the initial y velocity of the plane is negative as is the direction of the gravity vector.

The diagram below describes the situation:



The arrow shows the motion of the bomber at the moment of release. The x and y equations of motion are:

$$x(t) = v_{ox} t = v_o \sin 53^o$$

$$y(t) = y_o + v_{oy} t - \frac{1}{2} g t^2 = 800 \text{ m} - v_o \cos 53 t - \frac{1}{2} g t^2$$

a) In this problem, we know that the time of flight for the bomb is 5 s, so we can write :

$$y(5 \text{ s}) = 0 = 800 \text{ m} - v_o \cos 53^o (5 \text{ s}) - \frac{1}{2} g (5 \text{ s})^2$$

This yields a simple equation for v_o :

$$v_o = \frac{\left(800 \text{ m} - 0.5\left(9.8 \text{ m}/\text{s}^2\right)(5 \text{ s})^2\right)}{\cos 53^o (5 \text{ s})} = 225 \text{ m/s}$$

b) The horizontal distance traveled by the bomb is :

$$x = v_{ox} t = v_0 \sin 53^o (5 \text{ s}) = 225 \text{ m/s} \sin 53^o (5 \text{ s}) = 898 \text{ m}$$

c) Since there are no forces in the horizontal direction, the x component of velocity is the same throughout the trip. The y component of velocity varies according to :

$$v_y(t) = v_{oy} - g\,t = -225\cos 53^o\,m/s - \left(9.8\,m\big/s^2\right)(5\,s) = -184\,m/s$$

The magnitude of the final velocity is the Pythagorean sum of the two components :

$$|v| = \sqrt{(180\,m/s)^2 + (-184\,m/s)^2} = 257\,m/s$$

at an angle with respect to the ground of :

$$\theta = \tan^{-1}\left(\frac{|-184\,m/s|}{180\,m/s}\right) = 45.6$$

---

2. In this projectile motion problem, we are asked to determine whether a ball could clear a 7 m high fence located 95 m from the launch point. This means we want to know what the height of the ball will be when it is a certain distance downrange. This means that we want to transform our equations of motion from x (t) and y (t) to an equation that gives us y as a function of x, that is, we want to recast these equations as y (x). We begin with the standard equations of motion :

$$y(t) = y_o + v_{oy}\,t - \frac{1}{2}\,g\,t^2$$

$$x(t) = v_{o\,x}\,t$$

Our initial conditions are :

$$y_o = 1\,m;\ v_{oy} = v_o \sin 45;\ v_{ox} = v_o \cos 45$$

We can rewrite the x (t) equation as :

$$t = \frac{x}{v_o \cos\theta}\ \text{(I will call the angle } \theta \text{ and substitute numerical values at the end).}$$

Inserting this expression for t into the original y (t) equation gives us our expression for y (x) :

$$y(x) = y_o + v_o \sin\theta\left(\frac{x}{v_o\cos\theta}\right) - \frac{g\,x^2}{2\,v_o^2\cos^2\theta} = y_o + x\tan\theta - \frac{g\,x^2}{2\,v_o^2\cos^2\theta} \qquad (1)$$

Our ultimate goal will be to find the value of y when x = 95 m, but we cannot do that yet since we do not know the initial velocity. However, we do know that y = 0 when x = 105 m; we can subsitute these values in equation (1) above and solve for v_0 :

$$0 = 1\,m + 105\,m * \tan 45 - \frac{\left(9.8\,m\big/s^2\right)(105\,m)^2}{2\,v_0^2\cos^2 45}$$

Solving for v_o :

$$v_o = \sqrt{\frac{\left(9.8\,m\big/s^2\right)(105\,m)^2}{2\cos^2 45\,(106\,m)}} = 31.9\,m/s$$

Now, we substitute this value of the initial velocity into equation (1) to determine the height of the ball when it is a distance 95 m downrange :

$$y\,(95\,\mathrm{m}) = 1\,\mathrm{m} + 95\,\mathrm{m} * \tan 45 - \frac{\left(9.8\,\mathrm{m}/\mathrm{s}^2\right)(95\,\mathrm{m})^2}{2\,(31.9\,\mathrm{m}/\mathrm{s})^2 \cos^2 45} = 9.1\,\mathrm{m}$$

The ball will clear the fence.

---

3. We can treat the ball as a projectile that lands at the same level it is launched, so that we can write its equations of motion as :

$$y\,(t) = v_{oy}\,t - \frac{1}{2}\,g\,t^2 \qquad x\,(t) = v_{ox}\,t$$

We know from class work that we can write the time of flight and range for this object as :

$$t = \frac{2\,v_0 \sin\theta}{g} \qquad R = \frac{v_0^2 \sin(2\,\theta)}{g}$$

Using the values given we find :

$$t = \frac{2\,(20\,\mathrm{m}/\mathrm{s}) \sin 45}{9.8\,\mathrm{m}/\mathrm{s}^2} = 2.9\,\mathrm{s} \qquad R = \frac{(20\,\mathrm{m}/\mathrm{s})^2 \sin(90)}{9.8\,\mathrm{m}/\mathrm{s}^2} = 40.8\,\mathrm{m}$$

This means the other child has to run a distance of (55 m - 40.8 m) in 2.9 s, equivalent to a speed of 4.9 m/s.

---

4. The Earth' s average distance from the sun is $1.5 \cdot 10^{11}$m. The Earth makes one revolution in 1 year, so the tangential velocity of the Earth is:

$$v = \frac{2\,\pi\,R}{1\ \mathrm{year}} = \frac{2\,\pi\left(1.5 \times 10^{11}\,\mathrm{m}\right)}{365\ \mathrm{days}/\mathrm{yr} \times 24\ \mathrm{hrs}/\mathrm{day} \times 60\ \mathrm{min}/\mathrm{hr} \times 60\ \mathrm{sec}/\mathrm{min}} = \frac{2\,\pi\left(1.5 \times 10^{11}\,\mathrm{m}\right)}{3.15\,\mathrm{s}}$$

$$= 3 \times 10^4\,\mathrm{m}/\mathrm{s}$$

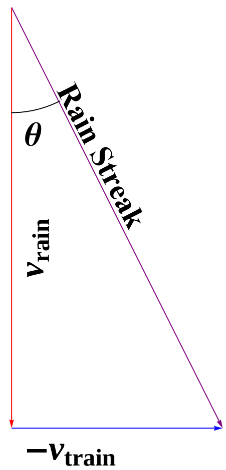
Thus, the centripetal acceleration toward the sun is :

$$a = \frac{v^2}{r} = \frac{\left(3 \times 10^4\,\mathrm{m}/\mathrm{s}\right)^2}{1.5 \times 10^{11}\,\mathrm{m}} = 6 \times 10^{-3}\,\mathrm{m}/\mathrm{s}^2$$

When you compare this to the acceleration of gravity on the Earth' s surface (9.81 m/s), you can understand why you do not feel the sensation of falling into the sun.

---

5. Let' s start with a diagram of the situation :



In this diagram, the red vertical line represents the velocity of the falling rain with respect to the ground, the blue horizontal line represents the motion of the ground with respect to the train, and the purple resultant is how the rain streak appears on the window of the train. From this diagram, we can see easily that the angle θ is described by :

$$\tan\theta = \frac{|v_{train}|}{v_{rain}} = \frac{V}{v}$$

In the notation of the book, we want to express the velocity of the rain with respect to the train in terms of the velocity of the rain with respect to the ground, and the velocity of the train with respect to the ground :

Using the notation of the book, we can express the velocity of the rain with respect to the train in terms of the velocity of the rain with respect to the ground and the velocity of the ground with respect to the train :

$$\mathbf{V}_{RT} = \mathbf{V}_{RG} + \mathbf{V}_{GT}$$

Remembering that velocities are vectors, we can write these vectors as :

$$\mathbf{V}_{RT} = v\,\hat{\mathbf{y}} - V\,\hat{\mathbf{x}}$$

where we adopt a coordinate system where down is positive (so the velocity of the rain is positive), and the velocity of the ground with respect to the train is negative since an observer on the train sees the ground moving backward.

---

6. This problem is similar to the problem where you analyzed the motion of a projectile launched from the edge of a tall building. This problem differs in that the ground is not level, but slopes downward at a 15 degree angle. Before writing equations, let' s think for a moment about which equations we want to solve.

We know that we will need to consider the x and y equations of motion for the arrow, but we also want to know when the arrow' s trajectory will intersect the sloping ground. Our strategy will be to convert the equations of motion to the y (x) equation (as we used in problem 2), and then write an equation for the sloping line. Our solution will require us to find that value of x that simultaneously satisfies the y (x) equation of motion and the equation of the sloping line. In other words, once we have our two equations, we equate them to find the value of x.

First, we refer back to problem 2 to express y as a function of x :

$$y(x) = y_o + x\tan\theta - \frac{g x^2}{2 v_0^2 \cos^2\theta}$$

Here, the initial value of y is 1.75 m and the initial velocity is 50 m/s. Now, we have to be careful about the value of θ since there are two angles involved in this problem. The angle we define as θ is the angle above the horizon that the arrow is launched. Here, θ = 20 degree.

Second, we write an equation for the sloping line. We know that all straight lines can be expressed as y = m x + b where m is the slope and b is the y - intercept. If we choose our coordinates such that the person is standing on the origin, the y - intercept of the sloping line is zero. The slope of this line is simply the tangent of the angle of the declivity, so we can write the equation of this line as :

$$y_{\text{line}} = -x\tan\phi$$

where ϕ is the slope angle. We include a minus sign since the slope is negative. The arrow will intersect the ground when these two equations are equal, or when :

$$y_o + x\tan\theta - \frac{g x^2}{2 v_0^2 \cos^2\theta} = -x\tan\phi$$

or :

$$\frac{g x^2}{2 v_0^2 \cos^2\theta} - (\tan\phi + \tan\theta)\,x - y_o = 0$$

This is a quadratic equation in x whose solution is:

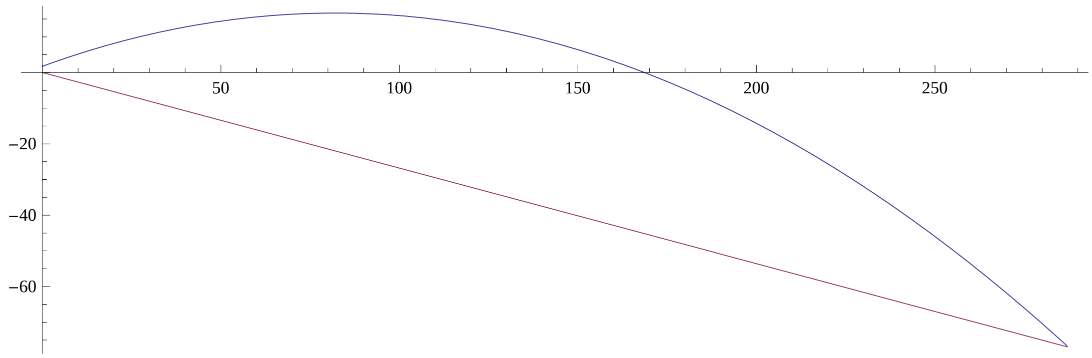
$$x = \frac{(\tan\theta + \tan\phi) \pm \sqrt{(\tan\theta + \tan\phi)^2 + \frac{2 g y_o}{v_0^2 \cos^2\theta}}}{\left(g / v_o^2 \cos^2\theta\right)}$$

Substituting values :

$$x = \frac{(\tan 15 + \tan 20) \pm \sqrt{(\tan 15 + \tan 20)^2 + \frac{2\left(9.8\,\text{m}/\text{s}^2\right)(1.75\,\text{m})}{(50\cos 20\,\text{m/s})^2}}}{\left(9.8\,\text{m}/\text{s}^2 / (50\cos 20\,\text{m}/\text{s})^2\right)}$$

$$\text{or}: \boxed{\mathbf{x = 287\,m}}$$

The diagram below shows the path of the arrow (blue curve) and the sloping hill (magenta?); the x axis represents where the level horizon would have been. Note that the arrow trajectory intersects the sloping hill when x = 287 m (the slope of the magenta line is 15^o)



7. In this problem, we realize that the horizontal motion of the projectile has two components : the horizontal component of v_o and the horizontal motion of the train, v_{train}. Since the train is moving along level ground, the vertical motion of the projectile is unaffected by the motion of the train. We will use a coordinate system where the launch point is at the origin, then with respect to an observer on the ground, the equations of motion are:

$$x(t) = (v_o \cos\theta + v_{\text{train}})\, t \tag{2}$$

$$y(t) = v_0 \sin\theta\, t - \frac{1}{2} g t^2 \tag{3}$$

To an observer on the ground, the horizontal motion of the projectile is $v_o \cos\theta + v_{\text{train}}$.

8. Now we are told that the train begins accelerating at the instant it launches the projectile, and we are asked to find the launch angle that will maximize the distance between the projectile and train at the moment of impact. To do this we will have to figure out the range of the projectile and how far the train moves while the projectile is in the air. We find the distance between projectile and train by subtracting these expressions. Finally, we use the methods of calculus to find the angle θ that will maximize this distance. To make my life simpler and allow me to avoid typing subscripts, I will call the initial launch velocity of the projectile v and the speed of the train V .

We find the range of the projectile using well known methods. First, we find the time of flight by setting eq. (3) equal to zero to obtain :

$$\text{time of flight} = \frac{2 v \sin\theta}{g}$$

Since the horizontal motion of the projectile is constant, its range (the distance it travels from its launch point) is its horizontal velocity multiplied by this time, or :

$$\text{range of projectile} = (v \cos\theta + V)\, t = (v \cos\theta + V)\,(2 v \sin\theta)/g$$

Now we find the distance traveled by the train during the time the projectile was in the air. Assuming the train started at $x_o = 0$, (and remember its initial velocity was V), the train' s equation of motion is :

$$x_{\text{train}}(t) = x_o + V t + \frac{1}{2} a t^2$$

But we know that t = 2 v sin θ/g, so using this expression for t in x (t) gives :

$$x_{train}(t) = V(2 v \sin\theta / g) + \frac{1}{2} a (2 v \sin\theta / g)^2$$

$$= \frac{2 v V \sin\theta}{g} + \frac{2 a v^2 \sin^2\theta}{g^2}$$

The distance between the projectile and train is :

$$\text{range} - x_{train}(t) = \frac{(v\cos\theta + V)(2 v \sin\theta)}{g} - \left(\frac{2 v V \sin\theta}{g} + \frac{2 a v^2 \sin^2\theta}{g^2}\right)$$

$$= \frac{\left(2 v V \sin\theta + 2 v^2 \sin\theta\cos\theta\right)}{g} - \left(\frac{2 v V \sin\theta}{g} + \frac{2 a v^2 \sin^2\theta}{g^2}\right)$$

Notice that the first terms in each parentheses cancel, leaving us with :

$$\text{range} - x_{train}(t) = \frac{2 v^2 \sin\theta\cos\theta}{g} - \frac{2 a v^2 \sin^2\theta}{g^2}$$

To find the value of θ that will maximize this distance, we have to differentiate this expression with respect to θ and set the derivative to zero. It will be a little easier if we rewrite the first term using the double angle formula (sin 2 θ = 2 sin θ cos θ) to get :

$$\text{range} - x_{train}(t) = \frac{v^2 \sin 2\theta}{g} - \frac{2 a v^2 \sin^2\theta}{g^2}$$

Now, taking the derivative with respect to θ :

$$\frac{d}{d\theta}\left(\frac{v^2 \sin 2\theta}{g} - \frac{2 a v^2 \sin^2\theta}{g^2}\right) = \frac{2 v^2 \cos 2\theta}{g} - \frac{4 a v^2 \sin\theta\cos\theta}{g^2}$$

Using the double angle formula to rewrite the last term on the right :

$$\frac{d}{d\theta}\left(\frac{v^2 \sin 2\theta}{g} - \frac{2 a v^2 \sin^2\theta}{g^2}\right) = \frac{2 v^2 \cos 2\theta}{g} - \frac{2 a v^2 \sin 2\theta}{g^2}$$

Now, to find the value of θ which maximizes the distance between objects, set this derivative to zero :

$$\frac{2 v^2 \cos 2\theta}{g} - \frac{2 a v^2 \sin 2\theta}{g^2} = 0 \Rightarrow \frac{2 v^2 \cos 2\theta}{g} = \frac{2 a v^2 \sin 2\theta}{g^2}$$

We can cancel out common factors of 2, v and g on both sides and get :

$$\cos 2\theta = \frac{a}{g} \sin 2\theta \Rightarrow \frac{\sin 2\theta}{\cos 2\theta} = \frac{g}{a}$$

Since tan x = sin x/cos x, this gives us :

$$\tan 2\theta = \frac{g}{a}$$

$$\boxed{\boldsymbol{\theta = \frac{1}{2} \tan^{-1}\left(\frac{g}{a}\right)}}$$

Let' s consider this result in a bit more detail. Whenever physicists derive a new result like this, we always compare the predictions of the new equation to what we expect would happen in specific (and often extreme) circumstances. For instance, what launch angle the equation predict if a = 0? If a = 0, the fraction (g/a) goes to infinity. This may seem impossible, but remember that the tan of 90 degree $\to \infty$. So our equation predicts that if the train is not accelerating, the projectile should be launched at a 45 degree (remember the 1/2 term) angle to maximize distance between the projectile and train. Does this make sense? (Isn' t 45 degree the angle we use to maximize distance on level ground?) Make sure you understand why this is the correct angle in this scenario when a = 0. What does the equation predict when a is very large? Can you explain why this result makes sense?