PHYS 111
HOMEWORK #5

Due : 29 Sept. 2016

This is a homework set about projectile motion, so we will be using the equations of motion throughout. Therefore, I will collect all those equations here at the start and reference them throughout the assignment.

$$x(t) = x_o + v_{ox}\, t + \frac{1}{2} a_x t^2 \text{ (if we ignore air friction, } a_x = 0) \tag{1}$$

$$y(t) = y_o + v_{oy}\, t - \frac{1}{2} g t^2 \text{ (assuming we set down as the negative direction)} \tag{2}$$

$$y(x) = y_o + x \tan\theta - \frac{g x^2}{2 v_0^2 \cos^2\theta} \text{ (assuming down is negative)} \tag{3}$$

$$v_x(t) = v_{ox} + a_x t \tag{4}$$

$$v_y(t) = v_{oy} - g t \text{ (assuming down is negative)} \tag{5}$$

$$v_f^2 = v_0^2 + 2 a s \text{ (s is distance traveled)} \tag{6}$$

It is permissible for you to reference these equations for your homework assignments, however, be prepared to derive any of all of these on exams.

1. A projectile is launched from level ground with an initial velocity of 30 m/s at an angle of 40^o with respect to the horizon. Determine:

a) the time of flight (5)

b) the range of the projectile (5)

c) the maximum height above the ground (5)

d) the components of v_x and v_y when the projectile strikes the ground (5)

e) the magnitude of the velocity and angle the velocity vector makes with respect to the ground just as the projectile lands (10)

Solutions : This is the absolutely standard projectile motion problem. For an object launched on level ground (so that the launch and final points are at the same height), we can set $x_o = 0$ as well as

$y_o = 0$.

a) We derived in class that the time of flight for a projectile on a level surface is:

$$t = \frac{2\, v_o \sin\theta}{g} = \frac{2\,(30\, m/s) \sin(40^o)}{9.8\, m/s^2} = 3.94\, s$$

b) Again, we derived in class that the range for a projectile on a level surface is :

$$R = \frac{2\, v_o^2 \sin\theta \cos\theta}{g} = \frac{2\,(30\, m/s)^2 \sin 40^o\ \cos 40^o}{9.8\, m/s^2} = 90.4\, m$$

c) The height above launch point is given by :

$$y_{max} = \frac{v_o^2 \sin^2\theta}{2\, g} = \frac{(30\, m/s)^2 (\sin 40^o)^2}{2 \cdot 9.8\, m/s^2} = 19\, m$$

d) We use equations (4) and (5) to find the final velocities. Since there are no horizontal forces, the horizontal component of velocity is constant. Therefore, the final horizontal velocity component is identical to the initial horizontal velocity, which is simply $v_o \cos\theta = 30$(m/s) $\cos 40^o$ = 23m/s.

The vertical component of velocity does vary since gravity acts in the vertical direction. Equation (5) leads us to:

$$v_{fy} = v_{oy} - g\, t$$

Where v_{fy} means the final y velocity and v_{oy} means initial y velocity. The initial y component of velocity is 30 m/s $\sin 40^o$ = 19.3m/s. Therefore, when the object hits the ground at t = 3.94s, its vertical velocity is:

$$v_{fy} = 19.3\, m/s - \left(9.8\, m/s^2\right)(3.94\, s) = -19.3\, m/s.$$

Is this a coincidence that the final vertical speed is equal to the initial vertical speed? Nope. In the absence of air friction, we should expect that an object has the same speed at launch and landing (assuming they occur at the same elevation.)

e) Now that we have the final horizontal and vertical components, we compute the final total speed and angle the velocity vector makes with the ground at the moment of impact.

We find the total speed by simply taking the Pythagorean sum of the components:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(23\, m/s)^2 + (19.3\, m/s)^2} = 30\, m/s$$

And the angle of the velocity vector with respect to the horizon at impact:

$$\tan\theta = \frac{v_y}{v_x} = \frac{19.3\, m/s}{23\, m/s} \Rightarrow \theta = \tan^{-1}(0.84) = 40^o$$

As expected, without air friction, the final parameters are the same as the launch parameters.

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2. Problem 14, p. 90.

Solution : There are a couple of ways you could have solved this problem, let' s try each of them in turn. First, let' s think about what we need to compute. We are asked to figure out how fast you have to run horizontally off of a cliff to make sure you miss the rocks at the bottom. Using the numbers for this problem, this means you need to be 1.75 m away from the cliff when you have fallen a distance of 9 m.

We are told the person runs horizontally off of the cliff; this means that she has no initial vertical motion. We also know that since there are no horizontal forces, that the horizontal velocity does not change during the flight.

We need to figure out what the horizontal velocity must be such that you have traveled 1.75 m by the time you fall a distance of 9 m.

First solution : We have previously shown (question 6/HW 4) that an object with no intial vertical motion will fall a distance H in the time:

$$t = \sqrt{2H/g}$$

Since the horizontal velocity never changes, the person will travel a horizontal distance given by:

$$x = v_x t = v_x \sqrt{2H/g}$$

in the time it takes to reach the ground. For our problem, H is 9m and x is 1.75 m. We can solve for the horizontal velocity:

$$v_x = \frac{x}{\sqrt{2H/g}} = \frac{1.75\text{ m}}{\sqrt{2(9\text{ m})/9.8\text{ m}/\text{s}^2}} = 1.29\text{ m}/\text{s}$$

Second solution: We do not have to compute the time at all. We could use equation (3) above. In this case, the launch angle is zero. If we choose a coordinate system in which down is positive, the person starts at $y_o = 0$ and ends at y = 9m, and g is a positive quantity. Then we have:

$$y(x) = y_o + x\tan\theta + \frac{g x^2}{2 v_o^2 \cos^2\theta}$$

(Remember, g is positive here.) Since $\theta = 0$, $\tan\theta = 0$ and $\cos\theta = 1$ and we have :

$$y(x) = \frac{g x^2}{2 v_o^2} \Rightarrow v_o = \sqrt{\frac{g x^2}{2 y(x)}}$$

Here, the final x value is 1.75m, and the final y value is 9 m (since we set the top of the cliff to y = 0). Thus we have:

$$v_o = \sqrt{\frac{9.8\ m/s^2\ (1.75\ m)^2}{2 \cdot 9\ m}} = 1.29\ m/s \text{ as before.}$$

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3. Problem 22, p. 91.

Solution : The first part of the problem asks you to figure out the initial speed of the grasshopper given certain information. The easiest way to do this is to make use of the maximum height and angle information. We know that the maximum height is determined by the initial vertical velocity of the object. In class we derived that the maximum height above the launch point is given by :

$$y_{max} = \frac{v_0^2 \sin^2\theta}{2\ g} \Rightarrow v_o = \sqrt{\frac{2\ g\ y_{max}}{\sin^2\theta}} = \sqrt{\frac{2\left(9.8\ m/s^2\right)(0.0674\ m)}{(\sin 50^o)^2}} = 1.5\ m/s$$

To find the height of the cliff, we can use Equation (3) from above (the time independent equation of motion. Let's call up the positive direction; then g is negative, the top of the cliff is at y = H and the bottom of the cliff is y = 0. We want to figure out the value of H, knowing the value of v_o and also that x = 1.06m when y = 0. We start with:

$$y(x) = y_o + x\tan\theta - \frac{g\ x^2}{2\ v_0^2\cos^2\theta} = H + (1.06\ m)\tan 50^o - \frac{\left(9.8\ m/s^2\right)(1.06\ m)^2}{2\ (1.5\ m/s)^2\cos^2 50^o}$$

Since y =0 when x = 1.06m, we can solve for H:

$$H = +\frac{\left(9.8\ m/s^2\right)(1.06\ m)^2}{2\ (1.5\ m/s)^2\cos^2 50^o} - (1.06\ m)\tan 50^o = 4.66\ m$$

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4. A projectile is launched from level ground with an initial velocity of v_o making an angle θ with the horizontal. At what angle must it be launched so that its range equals its maximum height?

Solution : For this problem, we make use of the relations derived in class for the range and maximum height of an object launched from a level surface. Our equations are :

$$\text{Range} = \frac{2\ v_0^2\sin\theta\cos\theta}{g} \qquad \text{Maximum height above launch point} = \frac{v_0^2\sin^2\theta}{2\ g}$$

We are asked to find the angle at which these are equal, so we equate these expressions and find:

$$\frac{2\ v_o^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin^2\theta}{2\ g}$$

Cancelling out common factors of v_o^2/g yields:

$$2\sin\theta\cos\theta = \frac{\sin^2\theta}{2}$$

As long as $\theta \neq 0$, we can divide through by $\sin\theta$ and get:

$$2\cos\theta = \frac{\sin\theta}{2} \Rightarrow \frac{\sin\theta}{\cos\theta} = 4 \text{ or } \tan\theta = 4 \Rightarrow \theta = 76^{o}$$

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5. A box slides along a (frictionless) table with speed v. The table is a height H above the ground. Determine expressions for the time the box will be in the air after sliding off the table, and how far the box lands from the edge of the table (find the horizontal distance from the edge of the table to the box' landing position).

Solution : This is exactly the same problem solved above in problem 2. Study those solutions carefully.

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6. Problem 40, p. 91

Solution : This problem combines several aspects of one and two dimensional motion. We also need to be a little mindful in choosing our coordinate system and our values for initial position.

Unlike most other trajectory problems, we are not given the launch velocity. Rather, we have to compute it from other information given. We can find the initial velocity from the statement that the rocket starts at rest and is accelerated along a 200 m ramp at a constant acceleration of 1.25 m/s^2 .

We can find its final speed via Equation (6):

$$v_f^2 = v_o^2 + 2\,a\,s$$

Where are told that the object starts from rest, so the initial velocity is zero. The rocket moves a distance s along the ramp (here, 200m), so we calculate the final velocity along the ramp:

$$v_f^2 = 2\left(1.25\,\text{m}/\text{s}^2\right)(200\,\text{m}) \Rightarrow v_f = \sqrt{500\,\text{m}^2/\text{s}^2} = 22.4\,\text{m}/\text{s}$$

This is the velocity of the rocket once it leaves the ramp and begins its free fall trajectory (free fall means the only force acting on the rocket is gravity).

Now that we have the initial velocity of the rocket, we can compute its trajectory. We are asked to find the maximum height above the ground and the range (with respect to the staring point of the motion at the base of the ramp.)

We have shown in class that the maximum height above the launch point is given by:

$$y_{max} = \frac{v_o^2 \sin^2\theta}{2\,g}$$

For the parameters here, $v_o = 22.4$m/s, $\theta = 35^o$, so

$$y_{max} = \frac{(22.4\ m/s)^2 (\sin 35^o)^2}{2\left(9.8\ m/s^2\right)} = 8.4\ m$$

Note carefully that this represents the height above the launch point at the end of the ramp. The question asks for maximum height above the ground. To find this, we need to add the height of the ramp, which we find simply from the diagram to be 200 sin 35^o = 114.7m. Therefore, the maximum height above the ground is 123.1m.

Now, as we find the range, we cannot use the simple range equation utilized in question 4. That range equation was derived assuming that the trajectory was on a level surface (i.e., the launch and landing points are on the same elevation). We write the equations of motion (Eqs. 1 and 2 from above). I will assign the coordinates $x_o = 0$ and y_o = 114.7m for the launch point at the end of the ramp. (Some of you might have called point A in the diagram the x = 0 point; either way is fine.). Then our equations of motion become:

$$x(t) = v_o \cos\theta\, t = (22.4 \cos 35^o\ m/s)\, t$$

$$y(t) = y_o + v_o \sin\theta\, t - \frac{1}{2} g t^2 = 114.7\ m + (22.4 \sin 35^o\ m/s)\, t - \left(4.9\ m/s^2\right) t^2$$

We need to find the time of flight; we find this by using the y(t) equation and solving for the time when y(t) = 0 (since y = 0 is the condition that the projectile has landed and the flight is over). Solving the quadratic:

$$t = \frac{-(12.8\ m/s) \pm \sqrt{(12.8\ m/s)^2 - 4\left(-4.9\ m/s^2\right)(114.7\ m)}}{-9.8\ m/s^2} = 6.3\ s$$

Knowing the time of flight is 6.3s, we use the x(t) equation to find the range of the rocket as:

$$x(t = 6.3\ s) = 18.3\ m/s \cdot 6.3\ s = 116\ m$$

Remember, this is 116 m from the launch point. The ramp has a horizontal run of (200 m) cos 35^o = 163.8m, so the final horizontal distance from the original point A is 116m+163.8m = 279.8m

We could also approach this problem using the time independent equation of motion (Equation 3 above):

$$y(x) = y_o + x \tan\theta - \frac{g x^2}{2 v_o^2 \cos^2\theta}$$

We wish to find the the value of x when y=0, so we solve the quadratic:

$$0 = 114.7\ m + x \tan 35^o - \frac{\left(9.8\ m/s^2\right) x^2}{2\,(22.4 \cos 35^o\ m/s)^2}$$

This is a quadratic in x, meaning solving this for x will give us the range from the launch point. Solving this yields two solutions, the positive one is 116 m as before.

7. Problem 53, p. 92

Solution : This is a trajectory problem in which the launch point (edge of the roof) and landing points are at different elevations. Therefore, we cannot use the simple range equation from problem 4, but must solve either the time dependent or time independent equations of motion.

Solving the time dependent equations (Eqs. 1 and 2) :

Since the initial velocity is downward, it is convenient to use down as the positive direction (we can just as easily set down as the negative direction, but I find this easier). If down is positive, the initial vertical velocity is positive as is gravity. The initial height is zero, and the ground is at y = 14 m.

Our initial conditions are then:

$$x_o = 0\,,\; y_o = 0;$$
$$v_{ox} = 7\cos 40^o\ \mathrm{m/s} = 5.4\ \mathrm{m/s};\; v_{oy} = 7\sin 40^o\ \mathrm{m/s} = 4.5\ \mathrm{m/s};$$

Using the y (t) equation of motion to find the time of travel:

$$y(t) = 14 = 0 + 4.5\,t + \frac{1}{2} g\,t^2 \Rightarrow \frac{1}{2} g\,t^2 + 4.5\,t - 14 = 0$$

The solutions of this quadratic are :

$$t = \frac{-4.5\ \mathrm{m/s} \pm \sqrt{(4.5\ \mathrm{m/s})^2 - 4\left(4.9\ \mathrm{m}/\mathrm{s}^2\right)(-14\ \mathrm{m})}}{9.8\ \mathrm{m}/\mathrm{s}^2} = 1.29\ \mathrm{s}$$

Since the x component of velocity is constant through the trip, the horizontal distance traveled is simply provided by:

$$x(t) = v_{ox}\,t \Rightarrow \text{range} = 5.4\ \mathrm{m/s} \cdot 1.29\ \mathrm{s} = 6.97\ \mathrm{m}$$

Using the time independent equation (Equation 3):

We use our same coordinate system (down is positive, initial y velocity is positive, g is positive, y = 0 at the edge of the roof and y = 14m on the ground). We want to know the value of x when y = 14m, and start with (notice that the sign of g is positive here):

$$y(x) = y_o + x\tan\theta + \frac{g\,x^2}{2\,v_o^2\cos^2\theta}$$

Using our initial and final values this becomes :

$$14\ \mathrm{m} = 0 + \tan(40^o)\,x + \frac{\left(9.8\ \mathrm{m}/\mathrm{s}^2\right)x^2}{2\,(5.4\ \mathrm{m/s})^2} = 0.84\,x + 0.168\,x^2$$

This yields the quadratic:

$$0.168\,x^2 + 0.84\,x - 14 = 0$$

Whose positive solution is 6.97m as above.

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8. Extra credit (this one is a little tougher, 15 points extra credit for a correct solution). An object is launched from the edge of a sheer cliff of height H at an angle θ with respect to the horizontal. If the object lands a distance D from the base of the cliff, show that its maximum height above the ground is :

$$H_{max} = H + \frac{D^2 \tan^2\theta}{4(H + D\tan\theta)}$$

Solution : We can use the diagram for problem 22 on p. 91 of your text as a reference (without of course, the numerical values). H is the height of the cliff above the ground level; H_{max} is the maximum height of the projectile above the ground. We have shown previously that the maximum height above the launch point is:

$$H' = \frac{v_0^2 \sin^2\theta}{2g} \tag{7}$$

I will use H' to represent the greatest elevation above the cliff, so that the greatest elevation with respect to the ground is simply H_{max} = H + H'

Our task now is to find an expression for H'.

Let's start by remembering that we can find expressions for the range of the projectile by using Eq. (3), the time independent equation of motion:

$$y(x) = y_o + x\tan\theta - \frac{g x^2}{2 v_0^2 \cos^2\theta}$$

where I am setting up as the positive direction; therefore g is negative, y_o = H and the final value of y = 0. We know that the x = D when y = 0, so our equation of motion yields:

$$0 = H + D\tan\theta - \frac{g D^2}{2 v_0^2 \cos^2\theta} \Rightarrow \frac{g D^2}{2 v_0^2 \cos^2\theta} = H + D\tan\theta \tag{8}$$

Your instinct might be to solve this quadratic equation for D, but that will give you an expression for D, not for H_{max} as we are requested.

Notice that our desired expression does not involve v_o in any way. This suggests to us that we have to use the information we have to eliminate v_o, by expressing v_o in terms of other variables.

Let's look back at Eq. (7), we can rewrite this to express v_o^2 as:

$$H' = \frac{v_0^2 \sin^2\theta}{2g} \Rightarrow v_o^2 = \frac{2gH'}{\sin^2\theta}$$

Now we take this expression for v_o^2 and substitute it into Eq. (8):

$$\frac{g\,D^2}{2\left(\frac{2\,g\,H'}{\sin^2\theta}\right)\cos^2\theta} = H + D\tan\theta$$

Rewriting the term on the left side:

$$\frac{g\,D^2\sin^2\theta}{4\,g\,H'\cos^2\theta} = H + D\tan\theta$$

Caqncelling out the common factors of g and recalling that $\tan\theta = \sin\theta / \cos\theta$, we have:

$$\frac{D^2\tan^2\theta}{4\,H'} = H + D\tan\theta$$

which allows us to write an expression for H’:

$$H' = \frac{D^2\tan^2\theta}{4\,(H + D\tan\theta)}$$

and since $H_{\max} = H + H'$, we can finally write:

$$H_{\max} = H + \frac{D^2\tan^2\theta}{4\,(H + D\tan\theta)} \quad \text{QED}$$