Fourier Series on other Intervals : Notes from Class

In class on W we investigated Fourier series on intervals other than (-π, π). For functions that are 2 L periodic (where 2 L is the wavelength), we compute Fourier coefficients and the Fourier series making the modifications derived in class :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n \pi x / L) + b_n \sum_{n=1}^{\infty} b_n \sin(n \pi x / L)$$

where :

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n \pi x / L)\, dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n \pi x / L)\, dx$$

Example 1 : Consider f (x) = x^2 on (-2, 2):

It is important to recall that 2L is the length of the interval, therefore 2 L = 2 - (-2) so that the value of L to use in the calculations is L = 2.

$$a_0 = \frac{1}{2} \int_{-2}^{2} x^2\, dx \qquad a_n = \frac{1}{2} \int_{-2}^{2} x^2 \cos(n \pi x / 2)\, dx \qquad b_n = \frac{1}{2} \int_{-2}^{2} x^2 \sin(n \pi x / 2)\, dx$$

Now, since f (x) is even on (-2, 2) we can employ symmetry to write :

$$a_0 = \frac{2}{2} \int_{0}^{2} x^2\, dx \qquad a_n = \frac{2}{2} \int_{0}^{2} x^2 \cos(n \pi x / 2)\, dx \qquad b_n = 0$$

Now, going the lazy route and letting Mathematica do the heavy lifting :

```
In[330]:= a0 = Integrate[x^2, {x, 0, 2}]
Out[330]= 8/3
In[332]:= an = Integrate[x^2 Cos[n π x / 2], {x, 0, 2}]
Out[332]= 8 (2 n π Cos[n π] + (-2 + n^2 π^2) Sin[n π]) / (n^3 π^3)
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and let' s just be sure :

```
In[337]:= bn = Integrate[x^2 Sin[n π x / 2], {x, -2, 2}]
Out[337]= 0
```

It is easy to show that the a coefficients reduce to :

$$a_n = \frac{16}{n^2 \pi^2} (-1)^n$$

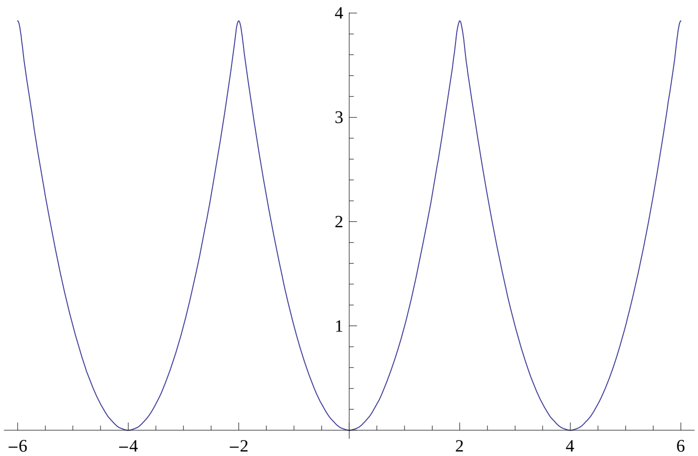
so that our Fourier series is :

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n \pi x / 2)}{n^2}$$

Verifying through Mathematica :

In[339]:=
```
Plot[4/3 + (16/π^2) Sum[(-1)^n Cos[n π x / 2] / n^2, {n, 1, 21}], {x, -6, 6}]
```

Out[339]=



Note that the values are consistent with our function (f (2) = 4) and the function repeats with a periodicity of 4.

Now, let' s examine the same function on (0, 4). Since our limits are not symmetric across the origin, we cannot make use of symmetry arguments. Our integrals become :

$$a0 = \frac{1}{2}\int_0^4 x^2\,dx \qquad an = \frac{1}{2}\int_0^4 x^2 \cos(n\pi x/2)\,dx \qquad b_n = \frac{1}{2}\int_0^4 x^2 \sin(n\pi x/2)\,dx$$

Even though the limits have changed, L = 2 since the total length of the interval, 2 L = 4.

Computing coefficients :

```
Clear[a0, an, bn, L, f]
```

In[340]:=
```
L = 2;
(* I know I have stressed we should not use capital letters for variables;
but I know that "L" is not restricted for Mathematica use, so let's use it. *)
f[x_] := x^2
a0 = (1 / L) Integrate[f[x], {x, 0, 4}]
an = (1 / L) Integrate[f[x] Cos[n π x / L], {x, 0, 4}]
bn = (1 / L) Integrate[f[x] Sin[n π x / L], {x, 0, 4}]
```

Out[342]= $\frac{32}{3}$

Out[343]= $\frac{8\left(2 n \pi \operatorname{Cos}[2 n \pi]+\left(-1+2 n^2 \pi^2\right) \operatorname{Sin}[2 n \pi]\right)}{n^3 \pi^3}$

Out[344]= $-\frac{8\left(1+\left(-1+2 n^2 \pi^2\right) \operatorname{Cos}[2 n \pi]-2 n \pi \operatorname{Sin}[2 n \pi]\right)}{n^3 \pi^3}$

And we get our three outputs. Knowing that sin (2 n π) is always zero for integer values of n, these complicated expressions reduce quite nicely, and we have :

$$a_n = \frac{16}{n^2 \pi^2} \qquad b_n = \frac{-16}{n \pi}$$

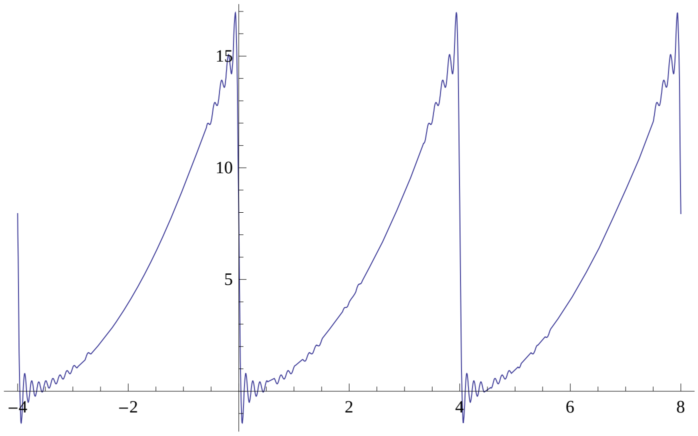
Our Fourier series then becomes :

$$f(x) = \frac{32}{6} + \sum_{n=1}^{\infty} \frac{16}{\pi^2} \frac{\cos(n\pi x/2)}{n^2} + \sum_{n=1}^{\infty} \left(\frac{-16}{\pi}\right) \frac{\sin(n\pi x/2)}{n^2}$$

Plotting three cycles of this function using the first 31 terms of the expansion :

In[345]:=
```
Plot[32 / 6 + Sum[(16 / π^2) Cos[n π x / 2] / n^2 + (-16 / π) Sin[n π x / 2] / n, {n, 1, 31}], {x, -4, 8}]
```

Out[345]=



And we reproduce our function over three cycles.