PRACTICE PROBLEMS FOR SUMMATION NOTATION

1. Evaluate :

a) $\epsilon_{ijk}\,\delta_{jk}$

Solution : There are two possible case, j = k and j ≠ k. If j = k, the ϵ term is zero and the whole product is zero. If j ≠ k, the δ is zero. In either case, the value of the product is zero.

b) $\epsilon_{jk2}\,\epsilon_{k2j}$

Solution : We can use the ϵ - δ relationship if we permute cyclically one set of indices :

$\epsilon_{jk2}\,\epsilon_{k2j} = \epsilon_{jk2}\,\epsilon_{jk2} = \delta_{kk}\,\delta_{22} - \delta_{k2}\,\delta_{2k} = 3\cdot 1 - \delta_{22} = 3 - 1 = 2$

2. Express in terms of Kronecker deltas :

a) $\epsilon_{ijk}\,\epsilon_{pjq}$ b) $\epsilon_{abc}\,\epsilon_{pqc}$

Solutions :

In a), we have a repeated index (j) in the same location, so we expand with respect to j :

$\epsilon_{ijk}\,\epsilon_{pjq} = \delta_{ip}\,\delta_{kq} - \delta_{iq}\,\delta_{kp}$

In b), expand with respect to c :

$\epsilon_{abc}\,\epsilon_{pqc} = \delta_{ap}\,\delta_{bq} - \delta_{aq}\,\delta_{cp}$

3. Prove the Jacobi identity (problem 14 on p. 284).

Solution : Expand each triple product according to the BAC - CAB rule :

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = [\mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})] + [\mathbf{C}\,(\mathbf{A}\cdot\mathbf{B}) - \mathbf{A}\,(\mathbf{B}\cdot\mathbf{C})] +$
$[\mathbf{A}\,(\mathbf{B}\cdot\mathbf{C}) - \mathbf{B}\,(\mathbf{C}\cdot.\mathbf{A})]$

I have color coded terms that cancel showing that the entire expression sums to zero.

4. Prove $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})\,(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})\,(\mathbf{B}\cdot\mathbf{C})$

Since we are taking the dot product of the vectors formed by A×B and C×D, we want to produce the i component of each vector, so we write :

$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) \rightarrow \epsilon_{ijk}\,A_j\,B_k\,\epsilon_{imn}\,C_m\,D_n = \epsilon_{ijk}\,\epsilon_{imn}\,A_j\,B_k\,C_m\,D_n$

Expand using the ϵ - δ relationship :

$\epsilon_{ijk}\,\epsilon_{imn}\,A_j\,B_k\,C_m\,D_n = \delta_{jm}\,\delta_{kn}\,A_j\,B_k\,C_m\,D_n - \delta_{jn}\,\delta_{km}\,A_j\,B_k\,C_m\,D_n$

In the first term on the rigth, j = m and k = n; in the second term, j = n and k = m. Making these substitutions :

$\delta_{jm}\,\delta_{kn}\,A_j\,B_k\,C_m\,D_n - \delta_{jn}\,\delta_{km}\,A_j\,B_k\,C_m\,D_n = A_m\,B_n\,C_m\,D_n - A_n\,B_m\,C_m\,D_n$

Grouping according to subscripts we obtain :

$A_m\,B_n\,C_m\,D_n - A_n\,B_m\,C_m\,D_n = (A_m\,C_m)\,(B_n\,D_n) - (B_m\,C_m)\,(A_n\,D_n)$

$= (\mathbf{A}\cdot\mathbf{C})\,(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})\,(\mathbf{B}\cdot\mathbf{C})$