FIRST GROUP WORK
29 JAN 2016

For this in class group assignment, you may use your text and the notes you have taken in class. You may not use any other sources nor may you access the internet. I will ask you to get as far on this assignment as you can in the next 20-25 minutes; each group will submit a written answer that I will use to assign the group grade. Make sure you put all students' names (first and last) on the sheet you submit. Send me your peer grades via email (using your Loyola email account) to me no later than 5 pm Monday. Remember the rules for submitting these grades: use only integer numbers from 0-10; the average of your grades must be 5 and you cannot give all students the same grade. Peer grades are absolutely confidential; do not discuss them with anyone in the class; any collusion in these grades will be considered academic dishonesty.

Consider the transformation :

$$x = a \cosh u \cos v$$

$$y = a \sinh u \sin v$$

where a is a constant and u, v are variables. Determine whether this transformation is orthogonal; find the scale factors h_u and h_v. If you have time, find expressions for the unit vectors $\hat{u}$ and $\hat{v}$.

If you are not familiar with cosh and sinh, these are called hyperbolic functions and have the properties :

$\frac{d}{dx} \cosh x = \sinh x$ $\qquad\qquad$ $\frac{d}{dx} \sinh x = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$

***Solution*s** : We begin by computing :

$$ds^2 = dx^2 + dy^2$$

Taking differentials :

$$dx = a \sinh u \cos v \, du - a \cosh u \sin v \, dv$$

$$dy = a \cosh u \sin v \, du + a \sinh u \cos v \, dv$$

Then :

$$\begin{aligned} ds^2 &= (a \sinh u \cos v \, du - a \cosh u \sin v \, dv)^2 + (a \cosh u \sin v \, du + a \sinh u \cos v \, dv)^2 \\ &= a^2 \left(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v\right) du^2 + a^2 \left(\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v\right) dv^2 \end{aligned}$$

Since the du dv terms cancel to zero, we can see this is an orthogonal transformation. The scale factors are then the square roots of each coefficient :

$$h_u = h_v = a\sqrt{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v}$$

We can simplify the argument of the radical by using the properties of the hyperbolic functions and write :

$$\cosh^2 u = 1 + \sinh^2 u$$

Then,

$$h_u = h_v = a\sqrt{\sinh^2 u \cos^2 v + \left(1 + \sinh^2 u\right)\sin^2 v}$$

$$= a\sqrt{\sinh^2 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 v} = a\sqrt{\sinh^2 u + \sin^2 v}$$

We can find the unit vectors by first writing the position vector as :

$$\mathbf{r} = a \cosh u \cos v\, \hat{\mathbf{x}} + a \sinh u \sin v\, \hat{\mathbf{y}}$$

$$\hat{\mathbf{u}} = \frac{\frac{\partial \mathbf{r}}{\partial u}}{\left|\frac{\partial \mathbf{r}}{\partial u}\right|} = \frac{a \sinh u \cos v\, \hat{\mathbf{x}} + a \cosh u \sin v\, \hat{\mathbf{y}}}{a\sqrt{\sinh^2 u + \sin^2 v}}$$

$$\hat{\mathbf{v}} = \frac{\frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial v}\right|} = \frac{-a \cosh u \sin v\, \hat{\mathbf{x}} + a \sinh u \cos v\, \hat{\mathbf{y}}}{a\sqrt{\sinh^2 u + \sin^2 v}}$$

We can also demonstrate the orthogonality of this transformation by showing that:

$\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \hat{\mathbf{v}} \cdot \hat{\mathbf{v}} = 1$ and $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = 0$