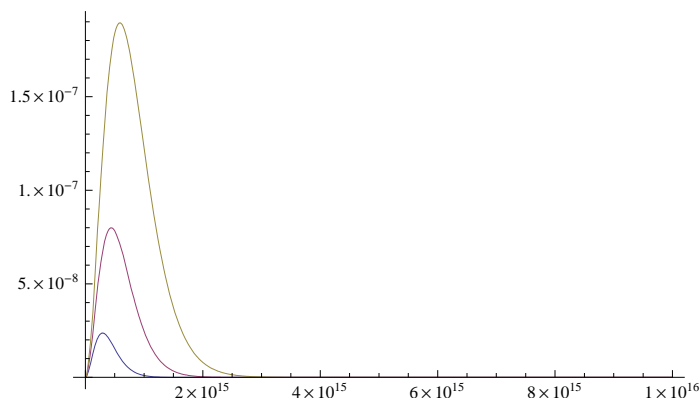


APPLYING MATHEMATICA FUNCTIONS TO THE PLANCK DISTRIBUTION LAW

In lab this past week, we used the Planck Distribution Law to study more advanced *Mathematica* functions. This distribution is :

$$B_{\nu}(T) d\nu = \frac{2 h \nu^3}{c^2} \frac{1}{e^{h \nu / k T} - 1} d\nu$$

where $B_{\nu}(T)$ is the amount of energy emitted by a blackbody radiation of temperature T per meter of surface per second per unit solid angle per frequency interval; h is Planck' s constant, k is Boltzmann' s constant, and c is the speed of light. If we plot this distribution law for three blackbody radiators of different temperatures, we observe curves of the form :



(the curves are for objects with temperatures of 10000 K (highest), 7500 K, 5000 K (lowest).

We can see that in each case there is a frequency at which the greatest amount of energy is emitted, and that the frequency of maximum energy emission is temperature dependent. We can compute an expression for this maximum (and this relationship is known as Wien' s Law) by differentiating the Planck distribution with respect to ν and setting the derivative to zero. Let' s do this via Mathematica :

```
In[44]:= Clear[b, ν, t, h, c, k]
```

```
b[ν_, t_] := (2 h ν^3 / c^2) (Exp[h ν / (k t)] - 1)^-1
```

```
D[b[ν, t], ν]
```

```
Out[46]= 
$$\frac{6 h \nu^2}{c^2 \left( -1 + e^{\frac{h \nu}{k t}} \right)} - \frac{2 e^{\frac{h \nu}{k t}} h^2 \nu^3}{c^2 \left( -1 + e^{\frac{h \nu}{k t}} \right)^2 k t}$$

```

To better understand what we will encounter next in Mathematica, note that if you set this equal to zero, you will get an equation :

$$3 = \frac{\frac{h\nu}{kT} e^{h\nu/kT}}{e^{h\nu/kT} - 1}$$

or :

$$3 = \frac{x e^x}{e^x - 1} \quad (1)$$

where $x = h\nu/kT$

Let's go back to the Planck distribution, and set the derivative to zero, and solve that equation in terms of ν :

```
In[47]:= Solve[D[b[ν, t], ν] == 0, ν]
```

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

```
Out[47]= {{ν -> \frac{k t (3 + ProductLog[-\frac{3}{e^3}])}{h}}}
```

and you get what looks like an error statement (but it isn't) and some answer that doesn't look right. But it is.

The Solve : ifun statement means that we are using an algebraic solving routine to find the solutions to a very non algebraic problem, and we might not have all the possible solutions that exist. However, we are obtaining the only real solution that exists. The solution that is generated produces a function that is new to us, the ProductLog function; this function helps in solving equations like eq. (1). In Mathematica, ProductLog[z] is the value of x that solves the equation :

$$z = x e^x$$

Let's try this out : suppose we want to find the value of x that satisfies

$$x e^x = 1$$

using ProductLog we find :

```
In[48]:= ProductLog[1] // N
```

```
Out[48]= 0.567143
```

Which we would also have found by using FindRoot :

```
In[49]:= FindRoot[x Exp[x] == 1, {x, 0}]
```

```
Out[49]= {x -> 0.567143}
```

So back to Planck. We could have avoided all this confusion with ProductLog (and this is what I omitted in lab on Monday) if we took the numerical value of our solution immediately, in other words :

```
In[50]:= Solve[D[b[v, t], v] == 0, v] // N
```

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

```
Out[50]= {{v -> \frac{2.82144 k t}{h}}}
```

and we obtain Wien's Law that is usually written :

$$\nu_{\max} = \frac{2.82 k T}{h}$$

where ν_{\max} is the frequency of the peak of the Planck curve, T is the absolute temperature, and h and k are the Planck and Boltzmann constant.

The total energy emitted by a blackbody of temperature T is found by integrating the distribution over all frequencies; doing so yields a remarkable result :

```
In[53]:= Integrate[b[v, t], {v, 0, \infty}, Assumptions -> Re[h / (k t)] > 0]
```

```
Out[53]= \frac{2 k^4 \pi^4 t^4}{15 c^2 h^3}
```

or the total energy emitted by a blackbody radiator is proportional to the fourth power of temperature. The cluster of constants is called the Stefan's constant, denoted by σ , and has an MKS value of 5.67×10^{-8} in MKS units. Therefore, if we look back at the Planck curves above, the hottest object emits 16 times more total energy than the coolest object depicted on the graph.