SERIES SOLUTION TO THE DIFFERENTIAL EQUATION DONE FOR GROUP WORK

$$y'' + x^2 y = 0$$

Assume a solution of the form: $y = \sum_{n=0} a_n x^n$

Substitute the trial solution into the differential equation:

$$\sum_{n=2} n(n-1)\, a_n x^{n-2} + \sum_{n=0} a_n x^{n+2}$$

Set k = n - 2 (so n = k+2) in the first sum; set k = n+2 (so n = k -2) in the second. Making these substitutions wherever there is an n yields:

$$\sum_{n=0} (n+2)(n+1)\, a_{n+2} x^n + \sum_{n=2} a_{n-2} x^n$$

Both sums have the same exponent, now we strip out the n = 0 and n =1 terms from the first sum to get:

$$2a_2 + 6a_3 x + \sum_{n=2} [(n+2)(n+1)\, a_{n+2} + a_{n-2}]\, x^n = 0$$

Since all the coefficients on the left must equal their analogues on the right, we know that:

$$a_2 = a_3 = 0 \quad \text{and} \quad a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)}$$

Because the (n+2) coefficient is related to the (n-2) coefficient, we know that $a_6 = a_{10}$ (= a_{14}, etc) since $a_2 = 0$. Similarly, all the coefficients for n = 7, 11, 15 are equal to zero since $a_3 = 0$.

Using the recursion relation we get:

$$a_4 = \frac{-a_o}{4 \cdot 3} \qquad a_8 = \frac{-a_4}{8 \cdot 7} = \frac{+a_o}{8 \cdot 7 \cdot 4 \cdot 3}$$

$$a_5 = \frac{-a_1}{5 \cdot 4} \qquad a_9 = \frac{-a_5}{9 \cdot 8} = \frac{+a_1}{9 \cdot 8 \cdot 5 \cdot 4}$$

Using these coefficients, we get for our power series:

$$y = a_o\left(1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \ldots\right) + a_1\left(x - \frac{x^5}{5 \cdot 4} + \frac{x^9}{9 \cdot 8 \cdot 5 \cdot 4} - \ldots\right)$$

By writing the solution this way, we see there are two independent solutions, one which is odd and one which is even.