PHYS 301
HOMEWORK #10-- Solutions

1. If your instinct is to grind away doing difficult integrals, this problem would be a great challenge. If however, you thought a bit more about the nature of line integrals, you would have noticed that the given function has a curl of zero. Therefore, the value of the line integral is path independent, and we can compute the line integral by choosing any path we wish. In particular, let' s choose the path along the x axis from - 3 to 3. Along this path, we have :

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_C \left(F_x\, dx + F_y\, dy\right)$$

Along the x axis, y = dy = 0, so this integral reduces to :

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_{-3}^{3} F_x\, dx = \int_{-3}^{3} e^x \cos y\, dx = e^3 - e^{-3} = 2\sinh 3$$

remember that along the x axis y = 0 so cos y = 1.

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2. We write Laplace's equation in spherical coordinates:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Since our given scalar :

$$V = r^n \cos\theta$$

has no ϕ dependence, we can set the last term on the right to zero, yielding :

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial (r^n \cos\theta)}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial (r^n \cos\theta)}{\partial\theta}\right) = 0$$

Recall that cos θ is a constant with respect to r, and r is a constant with respect to θ; doing the indicated differentiations yields :

$$\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}\left(r^2\left(n\, r^{n-1}\right)\right) + \frac{r^n}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin^2\theta\right) = 0$$

Differentiating again :

$$n\,(n+1)\cos\theta\, r^{n-2} + r^{n-2}\,(-2\cos\theta) = 0$$

Factoring out common terms :

$$r^{n-2}\cos\theta[n\,(n+1) - 2] = 0$$

Which yields the quadratic :

$$n^2 + n - 2 = 0 \Rightarrow n = 1, -2$$

3. $y'' + y = 0$

We already know that this is the simplest representation of harmonic motion; the solutions to this equation are sin and cos. Let' s use this to hone our skills with series solutions. Assume the trial solution :

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and substitute this into the original differential equation :

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

In the first sum, set k = n - 2 :

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + a_n x^n = 0$$

This yields the recursion relation :

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

Let' s notice a couple of things about this recursion relation. First, we notice a minus sign on the right, this tells us that terms will alternate in sign. Next, notice that the (n + 2) th term is a multiple of the nth term, indicating there will be an odd branch, and an even branch. Now, let' s evaluate coefficients :

$$a_2 = \frac{-a_0}{2\cdot 1} = \frac{-a_0}{2} \qquad a_3 = \frac{-a_1}{3\cdot 2}$$

$$a_4 = \frac{-a_2}{4\cdot 3} = \frac{-(-a_0)}{4\cdot 3\cdot 2} = \frac{a_0}{4!} \qquad a_5 = \frac{-a_3}{5\cdot 4} = \frac{-(-a_1)}{5\cdot 4\cdot 3\cdot 2} = \frac{a_1}{5!}$$

Our power series solution is :

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

Grouping terms gives us :

$$y = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right)$$

and we recognize these series immediately as :

$$y = a_0 \cos x + a_1 \sin x$$

We cannot provide values for the coefficients unless we are given initial conditions.

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4. $y'' + x\,y = 0$

Using our standard trial solution and subsituting into the original differential equation yields :

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} + \sum_{n=0}^{\infty} a_n\,x^{n+1} \;=\; 0$$

In the first sum, set k = n - 2, in the second sum set k = n + 1 and obtain :

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,x^n + \sum_{n=1}^{\infty} a_{n-1}\,x^n \;=0$$

We have to "strip out" the first term from the first sum so that both summations have the same limits :

$$2\,a_2 + \sum_{n=1}^{\infty} [(n+2)\,(n+1)\,a_{n+2} + a_{n-1}]\,x^n \;=\; 0$$

This tells us that :

$$a_2 \;=\; 0$$

and

$$a_{n+2} = \frac{-a_{n-1}}{(n+2)\,(n+1)}$$

This recursion relation tells us that :

$$a_3 = \frac{-a_0}{3\cdot 2} \qquad a_6 \;= \frac{-a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2}$$

$$a_4 = \frac{-a_1}{4\cdot 3} \qquad a_7 = \frac{-a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3}$$

$$a_5 = \frac{-a_2}{5\cdot 4} \;= 0 \qquad a_8 = \frac{-a_5}{8\cdot 7} = 0$$

and our general solution is :

$$y = a_0\left(1 - \frac{x^3}{6} + \frac{x^6}{180} - ...\right) + a_1\left(x - \frac{x^4}{12} + \frac{x^7}{504} - ...\right)$$

This differential equation is an example of the Airy equation, whose solution in closed form consists of Airy functions (useful in the theory of optics and quantum mechanics). Let' s see what Mathematica gives us if we try the DSolve feature :

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DSolve[y''[x] + x y[x] == 0, y[x], x]
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{{y[x] → AiryAi[(-1)^(1/3) x] C[1] + AiryBi[(-1)^(1/3) x] C[2]}}
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Does this equate to the series solution we obtained above? Let' s expand this solution as a series in x :

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Series[AiryAi[(-1)^(1/3) x] C[1] + AiryBi[(-1)^(1/3) x] C[2], {x, 0, 8}]
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$$\frac{3^{1/3}\,C[1]+3^{5/6}\,C[2]}{3\,\text{Gamma}\left[\frac{2}{3}\right]}-\frac{\left((-1)^{1/3}\left(3^{2/3}\,C[1]-3\times 3^{1/6}\,C[2]\right)\right)x}{3\,\text{Gamma}\left[\frac{1}{3}\right]}+\frac{\left(-3^{1/3}\,C[1]-3^{5/6}\,C[2]\right)x^3}{18\,\text{Gamma}\left[\frac{2}{3}\right]}+\frac{(-1)^{1/3}\left(3^{2/3}\,C[1]-3\times 3^{1/6}\,C[2]\right)x^4}{36\,\text{Gamma}\left[\frac{1}{3}\right]}+\frac{\left(3^{1/3}\,C[1]+3^{5/6}\,C[2]\right)x^6}{540\,\text{Gamma}\left[\frac{2}{3}\right]}-\frac{\left((-1)^{1/3}\left(3^{2/3}\,C[1]-3\times 3^{1/6}\,C[2]\right)\right)x^7}{1512\,\text{Gamma}\left[\frac{1}{3}\right]}+O[x]^9$$

There are lots of ugly constants, can we verify the two solutions are the same? Notice that both series expansions yield the same powers of x; that' s promising. Now, if we take ratios of coefficients, we see that they are consistent. In otherwords, notice that the ratio of the x^7 term is 1/504 of the x term, the ratio of the x^6 term is 1/180 of the x^0 term, and so on. By comparing ratios of coefficients, we get to ignore their inherent ugliness and verify that our series solution matches the closed form solution.

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5. $y'' - x^2 y = 0$

We know the routine by now, substitute our trial solution and get :

$$\sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-2} - \sum_{n=0}^{\infty} a_n\,x^{n+2} = 0$$

Set k = n - 2 in the first sum, k = n + 2 in the second :

$$\sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}\,x^n - \sum_{n=2}^{\infty} a_{n-2}\,x^n = 0$$

Now, in order to produce two sums with the same limits, we "strip out" the n = 0 and n = 1 terms from the first sum :

$$2\,a_2 + 6\,a_3\,x + \sum_{n=2}^{\infty} [(n+2)(n+1)\,a_{n+2} - a_{n-2}]\,x^n = 0$$

Since all the terms on the right must equal all the terms on the left, we know that :

$a_2 = a_3 = 0$

$$a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)}$$

Notice now there will not be alternating signs in the series expansion. Using the recursion relation we get :

$$a_4 = \frac{a_0}{4\cdot 3} \qquad a_8 = \frac{a_4}{8\cdot 7} = \frac{a_0}{8\cdot 7\cdot 4\cdot 3} = \frac{a_0}{672}$$

$$a_5 = \frac{a_1}{5\cdot 4} \qquad a_9 = \frac{a_5}{9\cdot 8} = \frac{a_1}{1440}$$

$$a_2 = a_{4n+2} = 0 \qquad a_3 = a_{4n+3} = 0$$

Our series solution is :

$$y = a_0\left(1 + \frac{x^4}{12} + \frac{x^8}{672} + ...\right) + a_1\left(x + \frac{x^5}{20} + \frac{x^9}{1440} + ...\right)$$

Let' s see what Mathematica says :

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DSolve[y''[x] - x^2 y[x] == 0, y[x], x]
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$$\left\{\left\{y[x] \to C[2]\,\text{ParabolicCylinderD}\left[-\frac{1}{2}, i\sqrt{2}\,x\right] + C[1]\,\text{ParabolicCylinderD}\left[-\frac{1}{2}, \sqrt{2}\,x\right]\right\}\right\}$$

$$\textbf{Series}\left[C[2]\,\text{ParabolicCylinderD}\left[-\frac{1}{2}, i\sqrt{2}\,x\right] + C[1]\,\text{ParabolicCylinderD}\left[-\frac{1}{2}, \sqrt{2}\,x\right], \{x, 0, 10\}\right]$$

$$\frac{2^{3/4}\sqrt{\pi}\,C[1] + 2^{3/4}\sqrt{\pi}\,C[2]}{2\,\text{Gamma}\left[\frac{3}{4}\right]} + \frac{\left(-2^{3/4}\sqrt{\pi}\,C[1] - i\,2^{3/4}\sqrt{\pi}\,C[2]\right)x}{\text{Gamma}\left[\frac{1}{4}\right]} +$$
$$\frac{\left(2^{3/4}\sqrt{\pi}\,C[1] + 2^{3/4}\sqrt{\pi}\,C[2]\right)x^4}{24\,\text{Gamma}\left[\frac{3}{4}\right]} + \frac{\left(-2^{3/4}\sqrt{\pi}\,C[1] - i\,2^{3/4}\sqrt{\pi}\,C[2]\right)x^5}{20\,\text{Gamma}\left[\frac{1}{4}\right]} +$$
$$\frac{\left(2^{3/4}\sqrt{\pi}\,C[1] + 2^{3/4}\sqrt{\pi}\,C[2]\right)x^8}{1344\,\text{Gamma}\left[\frac{3}{4}\right]} + \frac{\left(-2^{3/4}\sqrt{\pi}\,C[1] - i\,2^{3/4}\sqrt{\pi}\,C[2]\right)x^9}{1440\,\text{Gamma}\left[\frac{1}{4}\right]} + O[x]^{11}$$

Ugly^2. But notice that the series expansion has the same powers as our solution, and if you again take ratios of coefficients, you will see that the x^9 term is 1/1440 of the x term; the x^8term is 1/672 of the x^0 term, and in fact, our two series match.