## PHYS 301 <br> HOMEWORK \#11-- Solutions

1. We are asked to find the potential at point $O$ due to three charges along the $x$ axis. We use the principle of superposition to write the total potential as the sum of the individual potentials :

$$
\mathrm{V}_{\text {total }}=\mathrm{V}_{\mathrm{o}}+\mathrm{V}_{1}+\mathrm{V}_{2}
$$

where
$\mathrm{V}_{0}$ is the potential due to the charge at the origin
$\mathrm{V}_{1}, \mathrm{~V}_{2}$ are the potentials due to the charges
$\mathrm{r}_{1}$ and $\mathrm{r}_{2}$ from O
Since the potential due to a point charge is simply

$$
\mathrm{V}_{\text {point charge }}=\frac{\mathrm{Kq}}{\mathrm{~d}}
$$

where K is a constant, q is the charge and d is the distance of the charge from the observer, we can use the law of cosines to write the potential of this system as :

$$
\mathrm{V}=2 \frac{\mathrm{kq}}{\mathrm{r}}-\mathrm{kq}\left(\frac{1}{\mathrm{r}_{1}}+\frac{1}{\mathrm{r}_{2}}\right)
$$

Note here that the two charges away from the origin have the same sign (a different situation from the standard dipole we considered in class). Following the procedure developed in class, we can expand $1 / r_{1}$ and $1 / r_{2}$ in terms of Legendre polynomials and obtain:

$$
\mathrm{V}=2 \frac{\mathrm{kq}}{\mathrm{r}}-\frac{\mathrm{kq}}{\mathrm{r}}\left[\sum_{\mathrm{m}=0}^{\infty} \mathrm{P}_{\mathrm{m}}(\cos \theta)(\mathrm{a} / \mathrm{r})^{\mathrm{m}}+\sum_{\mathrm{m}=0}^{\infty}(-1)^{\mathrm{m}} \mathrm{P}_{\mathrm{m}}(\cos \theta)(\mathrm{a} / \mathrm{r})^{\mathrm{m}}\right]
$$

Now, notice that the two sums are added. If $m$ is even, the second term is positive and the two sums add. If $m$ is odd, the second term is negative and the two terms cancel. This means we can write the total potential as :

$$
\mathrm{V}=2 \frac{\mathrm{kq}}{\mathrm{r}}-2 \frac{\mathrm{kq}}{\mathrm{r}}\left[\sum_{\mathrm{m}=0, \text { even }}^{\infty} \mathrm{P}_{\mathrm{m}}(\cos \theta)(\mathrm{a} / \mathrm{r})^{\mathrm{m}}\right]
$$

Now, if we expand the sum we find the first few terms are :

$$
\mathrm{V}=2 \frac{\mathrm{kq}}{\mathrm{r}}\left[1-\left(\mathrm{P}_{\mathrm{o}}(\cos \theta)(\mathrm{a} / \mathrm{r})^{0}+\mathrm{P}_{2}(\cos \theta)(\mathrm{a} / \mathrm{r})^{2}+\mathrm{P}_{4}(\cos \theta)(\mathrm{a} / \mathrm{r})^{4}+\ldots\right)\right]
$$

We could further note that the first term in parentheses is simply equal to 1 , so that the lead term in the expansion is the $P_{2}$ term.

Finally, notice that the lead term in the expansion of the dipole moment was the $P_{1}$ term, the lead term in the quadrupole expansion is the $P_{2}$ term. What do you think is the lead term in the octupole expansion? In the monopole expansion?
2. In the case of linear air friction, we can write Newton' s second law as :

$$
\begin{gathered}
F_{x}=m a_{x}=m \frac{d v_{x}}{d t}=-k v_{x} \\
F_{y}=m a_{y}=m \frac{d v_{y}}{d t}=-k v_{y}-m g
\end{gathered}
$$

By choosing the negative sign for mg , we are establishing a coordinate system where all vectors acting up are positive, and all vectors acting down are negative. We can solve these equations easily to compute expressions for velocity and displacement. Let' s start with the x components :

$$
\mathrm{m} \frac{\mathrm{~d} \mathrm{v}_{\mathrm{x}}}{\mathrm{dt}}=-\mathrm{k} \mathrm{v}_{\mathrm{x}} \Rightarrow \frac{\mathrm{~d} \mathrm{v}_{\mathrm{x}}}{\mathrm{v}_{\mathrm{x}}}=\frac{-\mathrm{k}}{\mathrm{~m}} \mathrm{dt}
$$

Integrating both sides :

$$
\ln \mathrm{v}_{\mathrm{x}}=\frac{-\mathrm{kt}}{\mathrm{~m}}+\mathrm{C} \quad \text { (it is important to remember the constant of integration) }
$$

Exponentiating both sides :

$$
v_{x}=A e^{-k t / m}
$$

We find the value of the constant A by using the boundary condition that at $\mathrm{t}=0, v_{x}=v_{x}(0)=40$ $\cos 40^{\circ}$ so that $v_{x}(0)=\mathrm{A}$. Using this value in our $v_{x}$ expression yields:

$$
v_{x}(t)=v_{x}(0) e^{-k t / m}
$$

We find an expression for $\mathrm{x}(\mathrm{t})$ from :

$$
v_{x}=\frac{d x}{d t} \Rightarrow x=\int v_{x} d t=\int v_{x}(0) e^{-k t / m} d t=\frac{-m}{k} v_{x}(0) e^{-k t / m}+C
$$

here, the boundary condition is $\mathrm{x}(0)=0$, which yields :

$$
0=\frac{-\mathrm{m}}{\mathrm{k}} \mathrm{v}_{\mathrm{X}}(0)+\mathrm{C} \Rightarrow \mathrm{C}=\frac{\mathrm{m}}{\mathrm{k}} \mathrm{v}_{\mathrm{x}}(0)
$$

and the complete expression for $\mathrm{x}(\mathrm{t})$ is :

$$
x(t)=\frac{m}{k} v_{x}(0)\left(1-e^{-k t / m}\right)
$$

Before moving on to the $y$ components, let' s look at this for a moment and wonder if this expression reduces to the value we expect as $\mathrm{k}->0$. We know that neglecting friction, the range of a projectile is simply the original x component of velocity multiplied by the time of flight since there are no forces in the x direction to change that component of velocity. So, if we let k approach zero, what expression do we get for $\mathrm{x}(\mathrm{t})$, and does that expression approach our expected result? As k
approaches zero, we can expand the exponential term and if $\mathrm{kt} / \mathrm{m} \ll 1$ we can neglect all terms except the first two :

$$
\mathrm{x}(\mathrm{t})=\frac{\mathrm{m}}{\mathrm{k}} \mathrm{v}_{\mathrm{x}}(0)\left[1-\left(1-\frac{\mathrm{kt}}{\mathrm{~m}}+\ldots\right)\right]=\frac{\mathrm{m}}{\mathrm{k}} \mathrm{v}_{\mathrm{x}}(0)\left(\frac{\mathrm{kt}}{\mathrm{~m}}\right)=\mathrm{v}_{\mathrm{x}}(0) \mathrm{t}
$$

which is exactly what we expect in the zero friction case.

Now on to the y components:

$$
m \frac{d v_{y}}{d t}=-k v_{y}-m g \Rightarrow m \frac{d v_{y}}{k v_{y}+m g}=-d t
$$

Integrate both sides :

$$
\frac{\mathrm{m}}{\mathrm{k}} \ln \left|\mathrm{k} \mathrm{v}_{\mathrm{y}}+\mathrm{mg}\right|=-\mathrm{t}+\mathrm{C}
$$

Multiply through by $\mathrm{k} / \mathrm{m}$, exponentiate both sides and solve for $v_{y}$ :

$$
\mathrm{v}_{\mathrm{y}}(\mathrm{t})=\frac{1}{\mathrm{k}}\left(\mathrm{~A}^{-\mathrm{kt} / \mathrm{m}}-\mathrm{mg}\right)
$$

Applying the initial condition that at $\mathrm{t}=0 v_{y}=v_{y}(0)$, and rearranging algebraically yields a final expression for $v_{y}(\mathrm{t})$ :

$$
\mathrm{v}_{\mathrm{y}}(\mathrm{t})=\frac{1}{\mathrm{k}}\left[\left(\mathrm{k} \mathrm{v}_{\mathrm{y}}(0)+\mathrm{mg}\right) \mathrm{e}^{-\mathrm{kt} / \mathrm{m}}-\mathrm{mg}\right]
$$

To find $y(t)$ we integrate this expression for $v_{y}(t)$ and apply the initial condition that $y(0)=0$. While a little messy, this yields the expression:

$$
\mathrm{y}(\mathrm{t})=\frac{\mathrm{m}}{\mathrm{k}}\left(\mathrm{v}_{\mathrm{y}}(0)+\frac{\mathrm{mg}}{\mathrm{k}}\right)-\frac{\mathrm{e}^{-\mathrm{kt} / \mathrm{m}}}{\mathrm{k}}\left(\mathrm{v}_{\mathrm{y}}(0)+\frac{\mathrm{m}^{2} \mathrm{~g}}{\mathrm{k}}\right)-\frac{\mathrm{mgt}}{\mathrm{k}}
$$

There are many ways you could choose to factor this expression, and while this is not the most compact, it does show you the different terms we need to compute below.

Below is a short program that computes values for $\mathrm{y}(\mathrm{t}), v_{y}(\mathrm{t})$ and the range. Notice carefully how the program extracts specific solutions from an array of solutions, and then uses them to calculate values of interest.

Clear [vy0, m, g, k, t, z, c, vy, timeofflight, halftime, $x$, range, vx0, maxht]
(* Establish constants and initial conditions *)
$g=9.81 ; k=0.5 ; g=9.81 ; v y 0=40 \operatorname{Sin}[2 \pi / 9] ; m=1 ; v x 0=40 \operatorname{Cos}[2 \pi / 9] ;$
(*We know from our solutions that certain terms appear repeatedly in our expressions; to make the programming easier $I$ define several of those *)
$z\left[t \_\right]:=\operatorname{Exp}[-k t / m]$
(* $c$ is the constant of integration in the $y(t)$ equation *)
$c=m / k(v y 0+m g / k) ;$
(*I establish expressions for $v y(t), y(t)$ and $x(t)$ *)
vy[t_] := vy0z[t] $+m g / k z[t]-m g / k$
$y\left[t_{-}\right]:=-v y 0 z[t] / k-m^{\wedge} 2 g / k \wedge 2 z[t]-m g t / k+c$
$x\left[t \_\right]:=m v x 0 / k(1-z[t])$
(* Study carefully how $I$ use the "/." and [ [ ] ]notations to substitute a value obtained from Solve directly into a variable. Can you think of another problem where this technique might help? *)

```
timeofflight = t /. Solve[y[t] == 0, t][[2]];
halftime = t /. Solve[vy[t] == 0, t][[1]];
Print["The time of flight of this projectile = ", timeofflight, " secs"];
Print["The time to reach maximum height =", halftime, " secs"];
maxht = y[halftime];
range = x[timeofflight];
Print["The range of this projectile = ", range, " meters"]
Print["The maximum height achieved by this projectile is ", maxht, " meters"]
The time of flight of this projectile = 3.99355 secs
The time to reach maximum height =1.67491 secs
The range of this projectile = 52.9629 meters
The maximum height achieved by this projectile is 18.5614 meters
(* There were two Mathematica warnings that Solve may not have
    found all solutions to the equations, but I have deleted those *)
```

