PHYS 301
HOMEWORK #12

Solutions

1. Bessel functions of the first kind, denoted as $J_n(x)$ are solutions to Bessel's differential equation, and appear in many contexts including electromagnetic theory and diffraction theory. The generating function for these functions is:

$$g(x, t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)\, t^n \tag{1}$$

Use this generating function to establish these two recursion relations :

a) $J_{n-1}(x) + J_{n+1}(x) = \dfrac{2n}{x} J_n(x)$

Solution : We differentiate both sides of equation (1) with respect to t :

$$\frac{\partial g(x, t)}{\partial t} = \frac{\partial}{\partial t} e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \frac{\partial}{\partial t} \sum_{-\infty}^{\infty} J_n(x)\, t^n$$

Performing the partial derivatives :

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} \cdot \frac{x}{2} \cdot \left(1 + \frac{1}{t^2}\right) = \sum_{-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

(the limits of the summation don' t change since they are infinite). We recognize the exponential term as the generating function and rewrite the left hand side as :

$$\frac{x}{2} \cdot \left(1 + \frac{1}{t^2}\right) \sum_{-\infty}^{\infty} J_n(x)\, t^n = \sum_{-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

Multiplying the LHS :

$$\frac{x}{2} \sum_{-\infty}^{\infty} J_n(x)\, t^n + \frac{x}{2} \sum_{-\infty}^{\infty} J_n(x)\, t^{n-2} = \sum_{-\infty}^{\infty} n\, J_n(x)\, t^{n-1}$$

Now, if we want to equate, say, the t^3terms on both sides of the equation, we can set n =3 in the first equation, set n=5 in the second sum, and n=4 in the summation on the right. Multiply through by 2/x and we get for this term:

$$J_3\, t^3 + J_5\, t^3 = \frac{2 \cdot 4}{x}\, t^4$$

If now we re - index and set n = 4, divide by t^4 we get :

$$J_{n-1} + J_{n+1} = \frac{2\,n}{x}\,J_n$$

b) $J_{n-1}(x) - J_{n+1}(x) = 2\,J'_n(x)$

where $J'_n(x) = \frac{d(J_n(x))}{d\,x}$

Solution : For this problem, we differentiate with respect to x :

$$\frac{\partial}{\partial x}\,g(x,t) = \frac{\partial}{\partial x}\,e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \frac{\partial}{\partial x}\sum_{-\infty}^{\infty} J_n(x)\,t^n \Rightarrow$$

$$\frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J'_n(x)\,t^n$$

Substituting on the left:

$$\frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{-\infty}^{\infty} J_n(x)\,t^n = \sum_{-\infty}^{\infty} J'_n(x)\,t^n$$

Distribute terms on the left :

$$\frac{1}{2}\left[\sum_{-\infty}^{\infty} J_n(x)\,t^{n+1} - \sum_{-\infty}^{\infty} J_n(x)\,t^{n-1}\right] = \sum_{-\infty}^{\infty} J'_n(x)\,t^n$$

If we multiply through by 2 and equate powers of t on both sides of the equation, we obtain :

$$J_{n-1} - J_{n+1} = 2\,J_n'$$

c) Use the *Mathematica* BesselJ function to find expressions for for $J_{1/2}(x)$ and $J_{-1/2}(x)$. Then use the appropriate recursion relation to find an expression for $J_{3/2}(x)$ and compare your answer with the *Mathematica* expression.

Solution: We will use the identity in part a), and recognize that if n=1/2, we can write:

$$J_{-1/2} + J_{3/2} = \frac{2\,(1/2)}{x}\,J_{1/2} \Rightarrow J_{3/2} = \frac{J_{1/2}}{x} - J_{-1/2} \qquad (2)$$

Using Mathematica to find Bessel functions, we get :

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{BesselJ[1 / 2, x], BesselJ[-1 / 2, x]}
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$$\left\{\frac{\sqrt{\frac{2}{\pi}}\,\mathrm{Sin}[x]}{\sqrt{x}},\ \frac{\sqrt{\frac{2}{\pi}}\,\mathrm{Cos}[x]}{\sqrt{x}}\right\}$$

Using equation (2), we obtain :

$$J_{3/2} = \frac{\sqrt{\frac{2}{\pi x}}\ \mathrm{Sin}[x]}{x} - \sqrt{\frac{2}{\pi x}}\ \mathrm{Cos}[x]$$

Compare with the Mathematica result :

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BesselJ[3 / 2, x]
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$$\frac{\sqrt{\frac{2}{\pi}}\left(-\mathrm{Cos}[x] + \frac{\mathrm{Sin}[x]}{x}\right)}{\sqrt{x}}$$

and the results match.

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2. Find Legendre series for the following, writing out the first three non zero terms for each. You may leave your answers in terms of Legendre polynomials. (10 pts each part)

a) $f(x) = \begin{cases} x, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$

Solution : We expand the function as a Legendre series according to :

$f(x) = c_0 P_0 + c_1 P_1 + c_2 P_2 + \ldots$

where the coefficients are found from :

$$c_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$$

since this function is zero on (0, 1), our integrals become :

$$c_m = \frac{2m+1}{2} \int_{-1}^{0} x P_m(x)\, dx \Rightarrow$$

$$c_0 = \frac{1}{2} \int_{-1}^{0} x\, dx = \frac{-1}{4}$$

$$c_1 = \frac{3}{2} \int_{-1}^{0} x^2\, dx = \frac{1}{2}$$

$$c_2 = \frac{5}{2} \int_{-1}^{0} x \cdot \frac{1}{2}\left(3x^2 - 1\right) dx = \frac{-5}{16}$$

so that the Legendre polynomial is :

$$f(x) = \frac{-1}{4} P_o + \frac{1}{2} P_1 - \frac{5}{16} P_2 = \frac{-1}{4} + \frac{1}{2} x - \frac{5}{16} \frac{\left(3x^2 - 1\right)}{2}$$

b) $f(x) = 3x^2 + x - 1$

Finding the coefficients as before :

$$c_0 = \frac{1}{2}\int_{-1}^{1} 3x^2 + x - 1\,dx = 0$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} x\left(3x^2 + x - 1\right)dx = 1$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} \frac{\left(3x^2 - 1\right)}{2}\left(3x^2 + x - 1\right)dx = 2$$

and the Legendre series is :

$$f(x) = P_1 + 2P_2 = x + 2\cdot\frac{1}{2}\left(3x^2 - 1\right) = 3x^2 + x - 1$$

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This problem is very similar to the example in the book (and the problem we are doing in class). We are solving Laplace' s equation in Cartesian coordinates on a semi - infinite plane, so we know our solution will be :

$$T(x, y) = (A\cos kx + B\sin kx)\left(C e^{k y} + D e^{-k y}\right)$$

Since T must be finite as y grows large, we know C = 0. Since T = 0 when x = 0, we can conclude A = 0. The boundary condition that T = 0 at x = 10 requires that $k = n\,\pi/10$. We can combine all these results and write our solution as :

$$T(x, y) = B(\sin n\pi x/10)\, e^{-(n\pi y/10)}$$

We can see at this point that there is no single value of n that will satisfy this equation, but a sum of solutions will satisfy the equation, or

$$T(x, y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/10)\, e^{-n\pi y/10} \tag{3}$$

If we apply the lower boundary condition, namely that T (x, 0) = x, we get when we set y = 0 :

$$T(x, 0) = x = \sum_{n=1}^{\infty} B_n \sin(n\pi x/10) \tag{4}$$

The only remaining unknown is the set of B_n coefficients, but we realize that equation (4) simply the Fourier sine series for the odd function :

$$f(x) = x, \;\; -10 < x < 10$$

Finding the set of B_n coefficients will lead to a complete solution. We know that the Fourier sine coefficients for an odd function are given by :

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/10)\,dx$$

for L = 10 and f (x) = x, we have :

$$B_n = \frac{2}{10} \int_0^{10} x \sin(n \pi x / 10)\, dx = \frac{-20(-1)^n}{n \pi}$$

Substituting these coefficients into the solution, eq. (3) :

$$T(x, y) = \frac{20}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n \pi x / 10)\, e^{-n \pi y/10}}{n}$$