PHYS 301 HOMEWORK #1

Solutions

All assignments must be turned in at the beginning of class on the day they are due. No credit will be given to homework submitted after the solutions are posted. All answers must be accompanied by complete and clear work. Refer to the syllabus for the proper format for submitting homework.

You may use Mathematica to check your results, but your solutions must show all steps explicitly.

1. If m and n are positive integers, find the value of the integrals :

$$\int_{-\pi}^{\pi} \cos(nx) dx \qquad \int_{-\pi}^{\pi} \sin(mx) dx$$

Solution : These are elementary integrals :

$$\int_{-\pi}^{\pi} \cos(nx) \, dx = \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} = \frac{1}{n} (\sin(n\pi) - \sin(-n\pi)) = 0$$

An important result to remember throughout this semester is that $sin(n \pi) = 0$ for all integer values of n.

$$\int_{-\pi}^{\pi} \sin(mx) \, dx = \frac{-1}{n} \cos(mx) \Big|_{-\pi}^{\pi} = \frac{-1}{n} (\cos(n\pi) - \cos(-n\pi)) = 0$$

Since cos is an even function, we know that $\cos(x) = -\cos(-x)$. We could have deduced this result without ever evaluating the integral. Sin is an odd function, and if you integrate an odd function between limits that are symmetric across the origin (i.e., your range of integration is \pm L), the value of the integral is zero.

2. Use the sin and cos addition formulae to show that

 $\cos(n+m)x + \cos(n-m)x = 2\cos(nx)\cos(mx)$

Solution : We apply the addition formulae :

$$\cos(n+m)x = \cos(mx)\cos(nx) - \sin(mx)\sin(nx)$$
(1)

$$\cos(n-m)x = \cos(nx)\cos(-mx) - \sin(-mx)\sin(nx)$$
⁽²⁾

Since $\cos is$ even, $\cos (-m x) = \cos (m x)$; since $\sin is$ odd $\sin (-m x) = -\sin (m x)$. Equations (1) and (2) become :

$$\cos(n+m)x = \cos(mx)\cos(nx) - \sin(mx)\sin(nx)$$
(3)

$$\cos(n-m)x = \cos(nx)\cos(mx) + \sin(mx)\sin(nx)$$
(4)

Adding equations (3) and (4) yield the requested result.

3. Use the result of problem 2 to determine the value of

$$\int_{-\pi}^{\pi} \cos(n x) \cos(m x) dx$$

where n and m are integers. Make sure you consider separately the cases m = n and $m \neq n$

Solution:
$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = \frac{1}{2} \left[\int_{-\pi}^{\pi} \cos(n+m)x dx + \int_{-\pi}^{\pi} \cos(n-m)x dx \right] = \frac{1}{2} \left[\frac{\sin(n+m)x}{n+m} \Big|_{-\pi}^{\pi} + \frac{\sin(n-m)x}{n-m} \Big|_{-\pi}^{\pi} \right]$$

If n and m are integers (and not equal to each other), then (n + m) and (n - m) are also integers. We noted above that sin $(p \pi)$ is zero for all integer values of p. Therefore, the value of the integral above is zero when $n \neq m$.

When n = m, the original integral becomes :

$$\int_{-\pi}^{\pi} \cos^2(n x) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos(2 n x) \, dx = \pi)$$

(The equality follows from using the trig identities :

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 - (1 - \cos^2 x) \Rightarrow \cos^2 x = \frac{1}{2} (1 + \cos 2x)$$

4. The cost of producing two products, x and y, is given by :

$$C(x, y) = 3x^2 + 2xy + 2y^2$$

If 10 units of x and 5 units of y are being produced, and the rate of production of x is increasing by 2 units/day and the rate of production of y is decreasing by 1 unit/day, what is the instantaneous rate of change of C (x, y)?

Solution : In multivariable calculus, you learned the chain rule for partial derivatives :

$$\frac{dC}{dt} = \frac{\partial C}{\partial x}\frac{dx}{dt} + \frac{\partial C}{\partial y}\frac{dy}{dt} = (6x+2y)\frac{dx}{dt} + (2x+4y)\frac{dy}{dt}$$

If we evaluate this expression for the values given in the problem (x = 10, y = 5, dx/dt = 2 units/day, dy/dt = -1 unit/day) we get :

$$\frac{dC}{dt} = (60 + 10) (2 / day) + (20 + 20) (-1 / day) = 100 / day$$

5. The temperature distribution in a room is given by :

$$T(x, y, z) = z^2 + e^{(x^2 - y^2)}$$

where x, y and z are the familiar Cartesian coordinates. A particle moves through the room with a velocity vector given by $\mathbf{v} = \{3, -1, 0\}$. What is the instantaneous rate of change of temperature measured by the particle when it is at the point (0, 1, 2)?

Solution : We can use the chain rule again to write :

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

We can calculate the partial derivatives of T easily given the expression for T (x, y, z), and we recognize that dx/dt, dy/dt, and dz/dt are just the Cartesian coordinates of the velocity vector of the particle, in other words, {dx/dt, dy/dt, dz/dt} = {3, -1, 0}.

Therefore, taking partial derivatives we get :

$$\frac{dT}{dt} = 2 x e^{x^2 - y^2} \frac{dx}{dt} - 2 y e^{x^2 - y^2} \frac{dy}{dt} + 2 z \frac{dz}{dt}$$

Evaluating the expression at $\{0, 1, 2\}$ and knowing that dx/dt = 3, dy/dt = -1, and dz/dt = 0, we can compute :

$$\frac{dT}{dt} = 2(0) - 2(1)e^{-1} \cdot (-1) + 2(2) \cdot (0) = \frac{2}{e}$$