PHYS 301
HOMEWORK #2

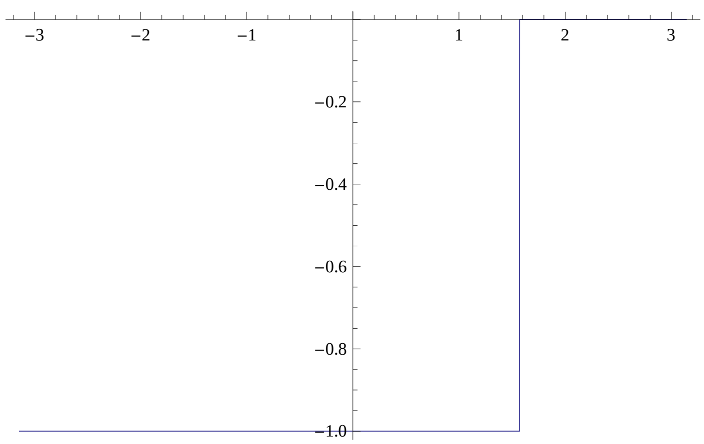
Solutions:

On this assignment, show solution to all integrals explicitly, although you may use symmetry arguments (but if you do, make sure you explain how you obtain your result).

Find the Fourier coefficients for the following functions, then write out explicitly the first three non - zero terms of each Fourier series (your answer should use the format shown in the solution to problem 5.3 on p. 354).

1. $f(x) = \begin{cases} -1, & -\pi < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$

Solution : First, it is a good idea to graph the function so we can more easily determine whether we can appeal to symmetry. This step function is :



The function is neither odd nor even. We compute Fourier coefficients (since the function is zero on (π/2, π) we only have to integrate between - π and π/2) :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi/2}(-1)\,dx = -3/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi/2}(-1)\cos(n x)\,dx = \frac{-1}{\pi}\left[\frac{1}{n}\sin(n x)\Big|_{-\pi}^{\pi/2}\right] = \frac{-1}{\pi n}[\sin(n\pi/2) - \sin(-\pi)]$$

We know that $\sin\pi = 0$, so we only need to evaluate the values of $\sin(n\,\pi/2)$. The table below shows the values of $\sin(n\,\pi/2)$ and the Fourier a coefficients as a function of n.

n	Sin (n π / 2)	a_n
1	1	$-1/\pi$
2	0	0
3	-1	$1/3\pi$
4	0	0
5	1	$-1/5\pi$

Now let' s compute the b_n coefficients :

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi/2}(-1)\sin(n\,x)\,dx = \frac{-1}{\pi}\left[\frac{-1}{n}\cos(n\,x)\Big|_{-\pi}^{\pi/2}\right] = \frac{1}{\pi\,n}[\cos(n\,\pi/2) - \cos(-n\,\pi)]$$

This yields an interesting set of coefficients. Since cos is an even function, cos (-n π) = cos (n π) and in class we learned that cos (n π) = $(-1)^n$. The table below shows values of cos(n π/2), cos(n π) and b_n:

n	cos (n π / 2)	cos (n π)	b_n
1	0	-1	$1/\pi$
2	-1	1	$-2/2\pi$
3	0	-1	$1/3\pi$
4	1	1	0
5	0	-1	$1/5\pi$

With these Fourier coefficients, we can write :

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + ... + b_1\sin x + b_2\sin 2x + b_3\sin 3x + ...$$

$$f(x) = \frac{-3}{4} - \frac{1}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + ...\right] + \frac{1}{\pi}\left[\sin x - \sin 2x + \frac{\sin 3x}{3} + ...\right]$$

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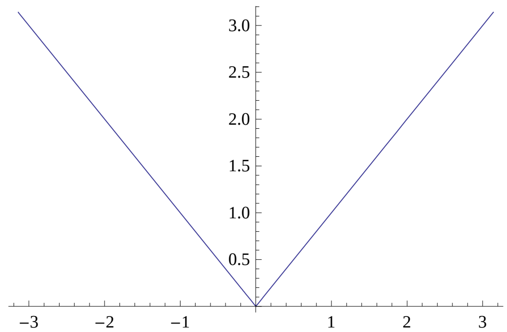
2. $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Solution : Plotting this graph reveals a very useful symmetry :



We recognize this as the graph of the even function Abs[x]. This means that we can apply symmetry arguments :

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(n x)\,dx \qquad b_n = 0$$

Remember that an even function (f (x)) times another even function (cos (n x)) yields an even function, and the product of an odd function (sin (n x) and an even function (f (x)) yields an odd function whose integral on (-π, π) is zero.

Then we have :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(n x)\,dx = \frac{2}{\pi}\left[\frac{x\sin(n x)}{n}\bigg|_o^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(n x)\,dx\right]$$

The first term in brackets (containing x sin (n x) is zero since sin is 0 at x = 0 and x = π), so our coeffiecients become :

$$a_n = \frac{-2}{\pi n}\int_0^{\pi}\sin(n x)\,dx = \frac{2}{\pi n}\left[\frac{1}{n}\cos(n x)\bigg|_0^{\pi}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{n even} \\ \frac{-4}{n^2\pi}, & \text{n odd} \end{cases}$$

and our Fourier series is :

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \ldots\right]$$

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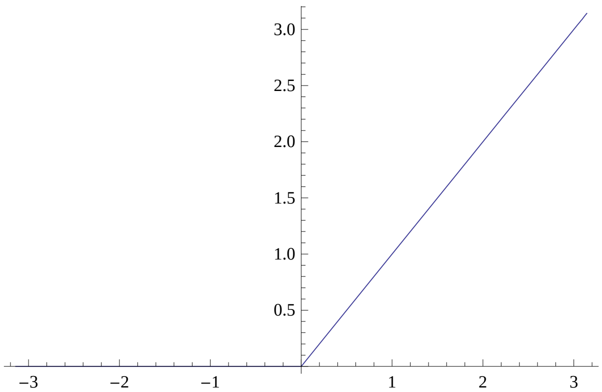
3. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Solution : We begin by plotting the function on (-π, π) :



We find the Fourier coefficients from :

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos(n\,x)\,dx = \begin{cases} 0, & \text{n even} \\ \frac{-2}{n^2\,\pi}, & \text{n odd} \end{cases}$$

(Note that we solved this integral in problem 2. These coefficients in problem 2 are twice since that function is even on (-π, π)).

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin(n\,x)\,dx = \frac{1}{\pi}\left[-\frac{x\cos(n\,x)}{n}\Big|_0^{\pi} - \left(\frac{-1}{n}\right)\int_0^{\pi}\cos(n\,x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n\,\pi)}{n} + \frac{1}{n^2}\sin(n\,x)\Big|_0^{\pi}\right] = \frac{-(-1)^n}{n}$$

This series has both a sin and cos series and the terms are :

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3\,x}{9} + \frac{\cos 5\,x}{25} + \ldots\right] + \left[\sin x - \frac{\sin 2\,x}{2} + \frac{\sin 3\,x}{3} - \ldots\right]$$

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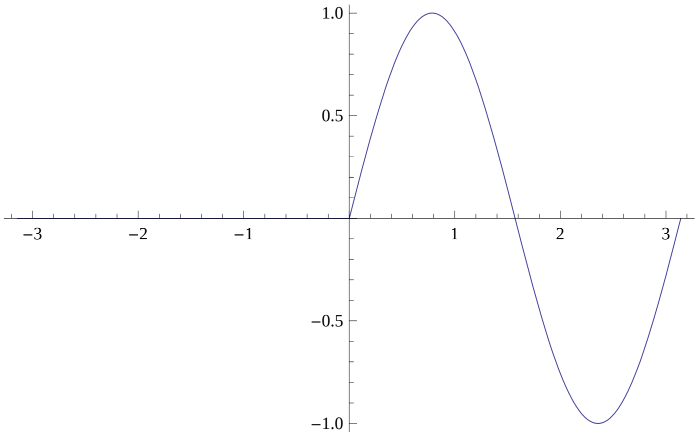
4. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin(2\,x), & 0 < x < \pi \end{cases}$

Solution : This function is neither odd nor even :



Now we find the Fourier coefficients. It is easy to compute a_0 directly by evaluating the integral, but if you look at the graph above, you should be able to intuit that the value of the integral is zero since there is as much area above the x axis as below. Thus, we can set $a_0 = 0$.

The integral for the a are reminiscent of problems 2 and 3 from the first homework set. To evaluate :

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin 2x \cos(n x)\, dx$$

we will need to rewrite the integrand in terms of the sums of other trig functions. Since we want terms involving the products of sin and cos, let' s see how we can use the sin addition formulae to help with this integral. Recall that :

$\sin(2+n)x = \sin 2x \cos(n x) + \sin(n x)\cos(2x)$

$\sin(2-n)x = \sin(2x)\cos(n x) - \sin(n x)\cos(2x)$

Adding these, we find that :

$$\sin(2x)\cos(n x) = \frac{1}{2}[\sin[(2+n)x] + \sin[(2-n)x]]$$

Using this result in our integral, we get :

$$a_n = \frac{1}{2\pi}\int_0^{\pi}[\sin[(2+n)x]+\sin[(2-n)x]]\,dx = \frac{1}{2\pi}\left[-\frac{\cos[(2+n)x]}{2+n}\Big|_0^{\pi} - \frac{\cos[(2-n)x]}{2-n}\Big|_0^{\pi}\right]$$

Now this gets a little messy. I will multiply through by - 1 and evaluate at the limits, yielding :

$$a_n = \frac{-1}{2\pi}\left[\frac{\cos[(2+n)\pi]-1}{2+n} + \frac{\cos[(2-n)\pi]-1}{2-n}\right]$$

As we evaluate this expression, remember that cos is 2π periodic, which means that $\cos(2\pi + n\pi) = \cos(n\pi)$, so the expression becomes :

$$a_n = \frac{-1}{2\pi}\left[\frac{(-1)^n-1}{2+n} + \frac{(-1)^n-1}{2-n}\right] = \frac{-1}{2\pi}\left[\frac{((-1)^n-1)((2+n)+(2-n))}{4-n^2}\right]$$

$$= \frac{1}{2\pi} \cdot \frac{1}{n^2-4}\left[4\left((-1)^n - 1\right)\right] = \begin{cases} 0, & \text{n even} \\ \frac{-4}{\pi}\,\frac{1}{n^2-4}, & \text{n odd} \end{cases}$$

Finding b_n: For these coefficients, we want to integrate terms of the form $\frac{1}{\pi}\int \sin 2x \sin(n\,x)\,dx$. We use the cos addition formulae:

$\cos(2-n)\,x = \cos 2\,x \cos(n\,x) + \sin 2\,x \sin(n\,x)$

$\cos(2+n)\,x = \cos 2\,x \cos(n\,x) - \sin 2\,x \sin(n\,x)$

We subtract these equations to show that :

$$\sin 2\,x \sin(n\,x) = \frac{1}{2}[\cos(2-n)\,x - \cos(2-n)\,x]$$

To find b_n we integrate:

$$b_n = \frac{1}{\pi}\int_0^\pi \frac{1}{2}[\cos[(2-n)\,x] - \cos[(2+n)\,x]]\,dx$$

This integral yields :

$$b_n = \frac{1}{2\pi}\left[\frac{\sin[(2-n)\,x]}{2-n}\Big|_0^\pi - \frac{\sin[(2+n)\,x]}{2+n}\Big|_0^\pi\right.$$

Your first instinct might be to look at these integrals and deduce that all the b_n are zero, since sin ($n\,\pi$) = 0 for all integer values of π. And that would be true for all n, except for the case where n = 2. In that case, notice that the first term on the right is indeterminate, The most straightforward way to find b_2, the only non-zero b coefficient, is to set n = 2 in the initial integral :

$$b_2 = \frac{1}{\pi}\int_0^\pi \sin 2\,x\ \sin 2\,x\ dx = \frac{1}{2}$$

Writing the first three non zero terms of this series :

$$f(x) = \frac{4}{\pi}\left[\frac{\cos x}{3\pi} - \frac{\cos 3\,x}{5} - \frac{\cos 5\,x}{21} - \frac{\cos 7\,x}{45} - \ldots + \frac{1}{2}\sin 2\,x\right]$$

Note that the first term in the cos series is positive, while the rest are negative. This occurs becasue of the n^2-4 term in the denominator.

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5. We determined in class (as does your text) the Fourier series for

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

Plot the first ten non - zero terms of the Fourier series for this function using Mathematica. Print out your output and attach it to your homework assignment; make sure you show your code along with your plot.

Solution : The coefficients for this series (see text, classnotes) are :

$$a_0 = 1 \qquad a_n = 0 \qquad b_n = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n\pi}, & n \text{ odd} \end{cases}$$

There are any number of ways you could write this code. The clunkiest (and klugiest) is to write out each term individually :

```
Plot[1 / 2 + (2 / π) (Sin[x] + Sin[3 x] / 3 + Sin[5 x] / 5 + Sin[7 x] / 7 + Sin[9 x] / 9 + Sin[11 x] / 11 +
    Sin[13 x] / 13 + Sin[15 x] / 15 + Sin[17 x] / 17 + Sin[19 x] / 19), {x, -π, π}]
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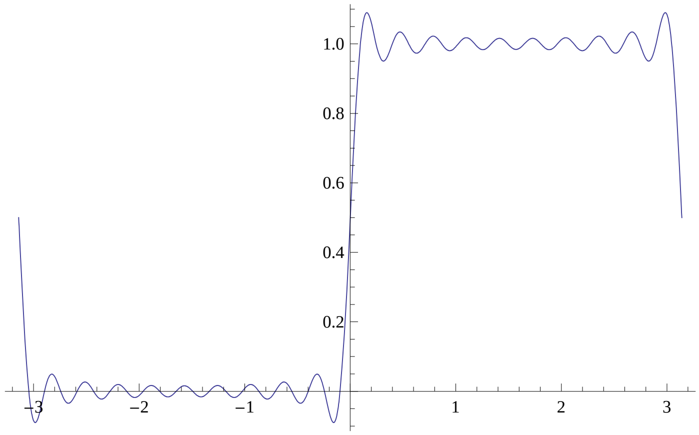

Ok, that works. Or we could go for the more compact :

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Plot[1 / 2 + (2 / π) Sum[Sin[n x] / n, {n, 1, 19, 2}], {x, -π, π}]
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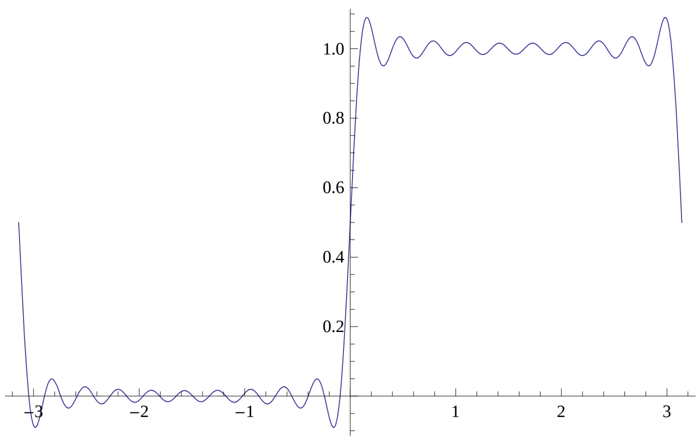

Or the slightly different :

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Plot[1 / 2 + (2 / π) Sum[Sin[(2 n - 1) x] / (2 n - 1), {n, 1, 10}], {x, -π, π}]
```