PHYS 301
HOMEWORK #3

Solutions

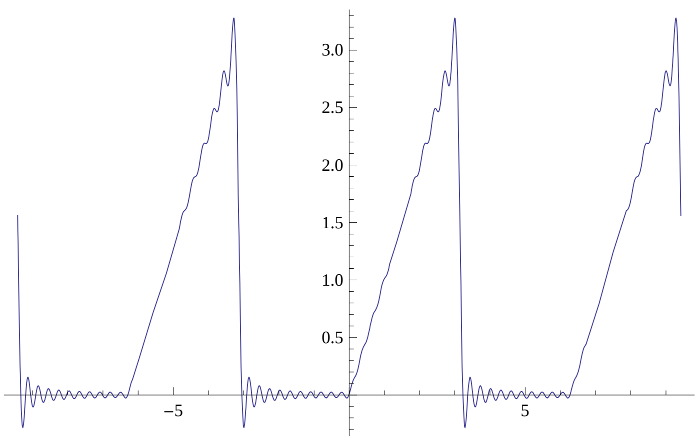
1. The function :

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

has the Fourier series :

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + ...\right] + \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ..\right]$$

We are asked to use Dirichlet' s Theorem to derive expressions for π using the values of f at x = 0, x = π/2 and x = π. Remember that Dirichlet' s theorem states that if a function can be expressed as a Fourier series, then the Fourier series converges to the value of f where f is continuous, and to the midpoint of a discontinuity. For this 2 π periodic function (I have plotted three cycles of this function) :



we see that f is continuous at x = 0 and x = π/2, but is discontinuous at x = π. Since f (0) = 0, we have :

$$f(0) = 0 = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos 0 + \frac{\cos 0}{9} + \frac{\cos 0}{25} + ...\right] + \left[\sin 0 - \frac{\sin 0}{2} + \frac{\sin 0}{3} - ...\right]$$

Since cos 0 = 1 and sin 0 = 0, we obtain :

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{9} + \frac{1}{25} + ...\right] \Rightarrow \frac{\pi^2}{8} = \sum_{n\ \mathrm{odd}}^{\infty} \frac{1}{n^2}$$

Let' s check this with Mathematica :

```
Sum[1/n^2,{n,1,∞,2}]
```

$\frac{\pi^2}{8}$

Now, at x = π/2, f (x) = π/2 and the series converges to this value, so :

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos\frac{\pi}{2} + \frac{\cos(3\pi/2)}{9} + \frac{\cos(5\pi/2)}{25} + ...\right] + \left[\sin\pi/2 - \frac{\sin(\pi)}{2} + \frac{\sin(3\pi/2)}{3} + ...\right]$$

In this case, all the cos terms are zero since cos of π/2, 3 π/2, 5 π/2, etc is zero. The sin terms vanish for even values of n, and the sign of the odd terms alternate, so we have :

$$\frac{\pi}{2} = \frac{\pi}{4} + \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right] \Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

Verifying through Mathematica :

```
Sum[-(-1)^n Sin[n π/2]/n,{n,1,∞,2}]
```

$\frac{\pi}{4}$

Finally, at x = π we have to realize that there is a discontinuity whose midpoint is π/2, therefore, we have :

$$f(\pi) = \frac{\pi}{2} = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos\pi + \frac{\cos 3\pi}{9} + \frac{\cos 5\pi}{25} + ...\right] + [\text{a series of } \sin(n\pi) \text{ terms}]$$

we know the sin (n π) terms are all zero, and that cos (n π) = -1 for all odd values of n, so Dirichlet' s theorem gives us :

$$\frac{\pi}{2} = \frac{\pi}{4} - \frac{2}{\pi}\left[-1 - \frac{1}{3} - \frac{1}{5} - ...\right] \Rightarrow \frac{\pi^2}{8} = \sum_{\text{odd } n}^{\infty} \frac{1}{n^2}$$

as we found in part a).

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2. We are asked to find two Fourier series. In both cases, the function is 2 L periodic on the given interval. Since the function is 2 L periodic, we do not need to extend the function (as we will in problems from section 9). We are asked to find the Fourier series for f (x) = 1 + x on (0, 4) and then on (-2, 2). In both cases, the interval is 4, so 2 L = 4 and L = 2. To find the Fourier coefficients (and then the series) on (0, 4), we write :

```
a0 = 1/2 Integrate[1+x,{x,0,4}]
```

6

Since the interval is (0, 4), we cannot make use of symmetry arguments. For the other coefficients we have :

$$a_n = \frac{1}{2}\int_0^4 (1 + x)\cos(n\pi x/2)\,dx = 0$$

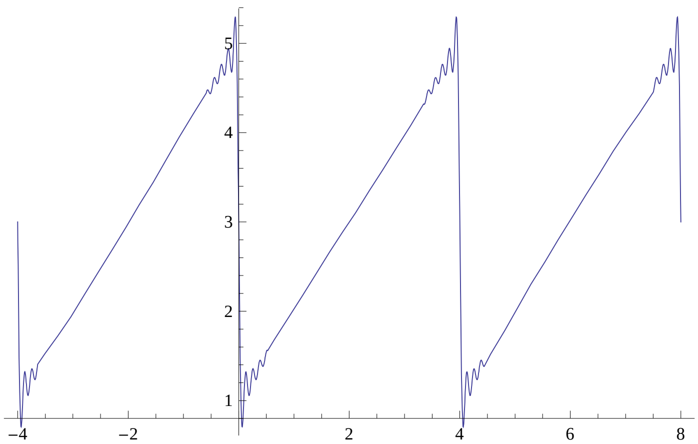
$$b_n = \frac{1}{2}\int_0^4 (1+x)\sin(n\pi x/2)\,dx = \frac{-4}{n\pi}$$

so that the Fourier expansion becomes :

$$f(x) = 3 - \frac{4}{\pi}\left[\sin(\pi x/2) + \frac{\sin(\pi x)}{2} + \frac{\sin(3\pi x/2)}{3} + \ldots\right]$$

And plotting this series over three cycles :

```
Plot[3 - (4 / π) Sum[Sin[n π x / 2] / n, {n, 1, 31}], {x, -4, 8}]
```



For part b), our value of L = 2 still, but our limits of integration are between (-2, 2), so we obtain :

$$a0 = \frac{1}{2}\int_{-2}^{2}(1+x)\,dx = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2}(1+x)\cos(n\pi x/2)\,dx$$

Even though f (x) is neither even nor odd, we can break the function into an even piece (here, simply 1) and an odd piece (x). We know the integral of x cos (n π x/2) will be zero on (-2, 2) since the integrand is odd. Therefore, the only non - zero part of this integral is :

$$a_n = \frac{1}{2}\int_{-2}^{2} 1\cdot\cos(n\pi x/2)\,dx = \frac{2}{2}\int_0^2 \cos(n\pi x/2)\,dx = \frac{2}{n\pi}\sin(n\pi x/2)\Big|_0^2 = 0$$

Computing the b coefficients (and employing symmetry to simplify the integral) :

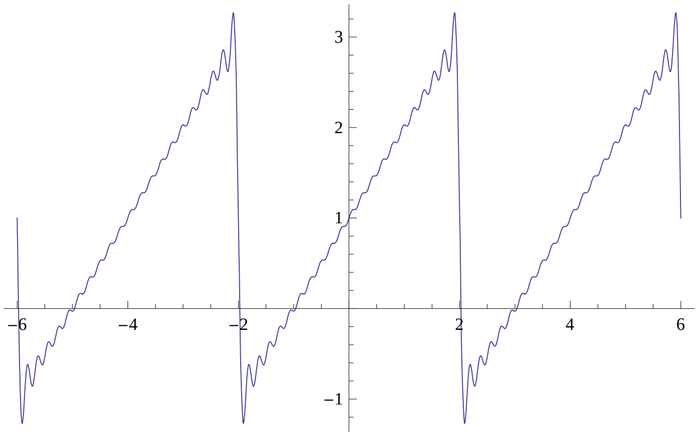
$$b_n = \frac{1}{2}\int_{-2}^{2}(1+x)\sin(n\pi x/2)\,dx = \frac{2}{2}\int_0^2 x\sin(n\pi x/2)\,dx = \frac{-4(-1)^n}{n\pi}$$

Our Fourier series is :

$$f(x) = 1 + \frac{4}{\pi}\left[\sin(\pi x/2) - \frac{\sin(\pi x)}{2} + \frac{\sin(3\pi x/2)}{3} - \ldots\right]$$

Plotting three cycles :

```
Plot[1 - (4 / π) Sum[(-1)^n Sin[n π x / 2] / n, {n, 1, 21}], {x, -6, 6}]
```



3. Find the Fourier series for f (x) = x

a) on $(-\pi, \pi)$:

We can make use of symmetry here since the function is odd. Therefore,

$$a_o = a_n = 0$$

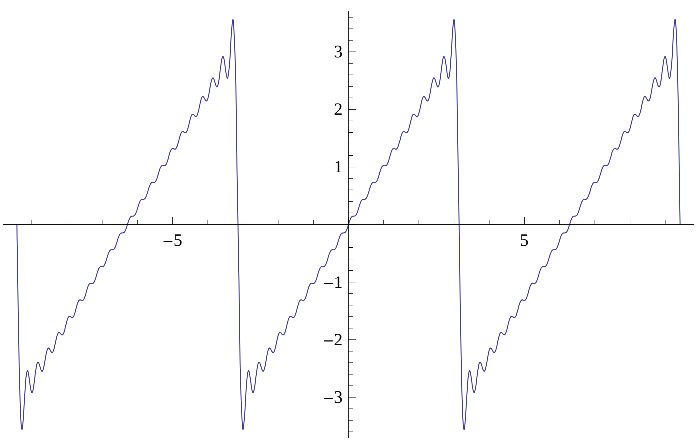
$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin(n x)\, dx = -2(-1)^n / n$$

And our series is :

$$f(x) = 2\left[\sin x - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \ldots\right]$$

Plotting three cycles :

```
Plot[-2 Sum[(-1)^n Sin[n x] / n, {n, 1, 21}], {x, -3 π, 3 π}]
```



b) on (-3, 3) We can still use symmetry, but must remember to use the proper form of the Fourier coefficients and series :

Because of the odd symmetry of the function, all the a coefficients are zero, and :

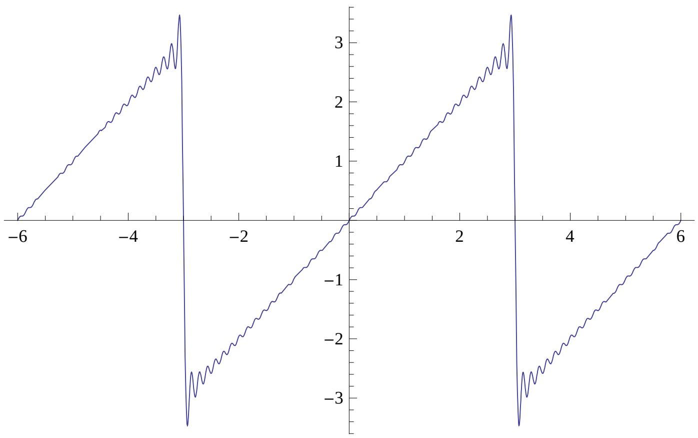
$$b_n = \frac{2}{3}\int_0^{3} x \sin(n \pi x / 3)\, dx = \frac{-6(-1)^n}{n \pi}$$

The Fourier series is :

$$f(x) = \frac{6}{\pi}\left[\sin(\pi x/3) - \frac{\sin(2\pi x/3)}{2} + \frac{\sin(3\pi x/3)}{3} + \ldots\right]$$

Plotting three cycles :

```
Plot[(-6 / π) Sum[(-1)^n Sin[n π x / 3] / n, {n, 1, 41}], {x, -6, 6}]
```



4. $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 3 \end{cases}$

This function is 2 L periodic on (-1, 3). This means that L = 2. To find our Fourier coefficients, we compute :

$$a0 = \frac{1}{2}\int_{-1}^{3} f(x)\,dx = \frac{1}{2}\left[\int_0^3 dx + \int_{-1}^{0}(-1)\,dx\right] = 1$$

$$a_n = \frac{1}{2}\int_{-1}^{3} f(x)\cos(n\pi x/2)\,dx = \frac{1}{2}\left[\int_0^3 \cos(n\pi x/2)\,dx + \int_{-1}^{0}(-1)\cos(n\pi x/2)\right]$$

$$= \frac{1}{n\pi}(\sin(3n\pi/2) - \sin(n\pi/2)) = \begin{cases} 0, & \text{n even} \\ \frac{-2}{n\pi}, & n = 1, 5, 9, \ldots \\ \frac{2}{n\pi}, & n = 3, 7, 11, \ldots \end{cases}$$

$$b_n = \frac{1}{2}\left[\int_0^3 \sin(n\pi x/2)\,dx + \int_{-1}^{0}(-1)\sin(n\pi x/2)\,dx\right]$$

$$= \frac{1}{2}\left[\frac{-2}{n\pi}\cos(n\pi x/2)\Big|_0^3 + \frac{2}{n\pi}\cos(n\pi x/2)\Big|_{-1}^{0}\right]$$

$$= \frac{1}{n\pi}[-(\cos(3n\pi/2) - 1) + (1 - \cos(n\pi/2))]$$

$$= \frac{1}{n\pi}[2 - \cos(3n\pi/2) - \cos(n\pi/2)] = \begin{cases} \frac{2}{n\pi}, & \text{n odd} \\ \frac{4}{n\pi}, & n = 2, 6, 10, \ldots \\ 0, & n = 4, 8, 12, \ldots \end{cases}$$

The Fourier series is :

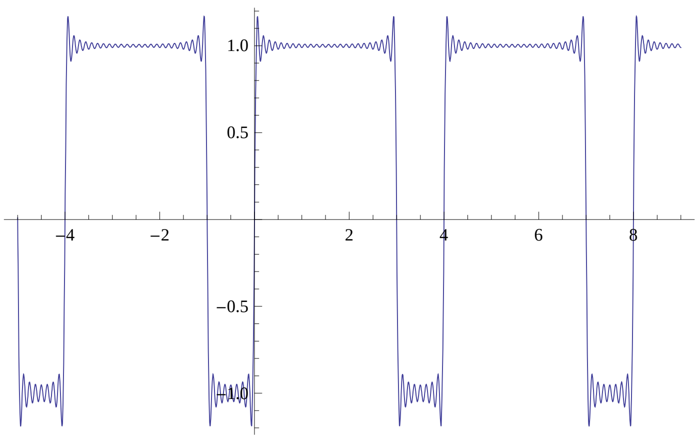
$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\cos(\pi x/2) - \frac{\cos(3\pi x/2)}{3} + \frac{\cos(5\pi x/2)}{5} - \ldots\right] +$$

$$\frac{2}{\pi}\left[\sin(\pi x/2) + \frac{2\sin(2\pi x/2)}{2} + \frac{\sin(3\pi x/2)}{3} + \frac{\sin(5\pi x/2)}{5} + 2\,\frac{\sin(6\pi x/2)}{6} + \ldots\right]$$

5. Plot the Fourier series from problem 4 :

In[42]:=
```
Plot[1/2 + (1/π) Sum[(Sin[3 n π / 2] - Sin[n π / 2]) Cos[n π x / 2] / n +
    (2 - Cos[3 n π / 2] - Cos[n π / 2]) Sin[n π x / 2] / n, {n, 1, 31}], {x, -5, 9}]
```

Out[42]=



6. Every three months, the existing amount of money receives 1 % interest; after the interest is applied, you deposit $100.

```
(* Do Loop *)
Clear[money, interest,deposit]
money=1000.0;deposit=100;
interest=0.01;
Do[money=(1+interest) money+deposit,{n,40}]
Print["At the end of ten years you will have $",money," in the bank"]
```

```
At the end of ten year you will have $6377.5 in the bank
```

Each value of n represents a three month period; in 10 years there are 120 months and so 40 accumulation periods.

In[109]:=
```
(* For statement *)
Clear[money,interest,deposit]
deposit=100;interest=0.01;
For[money=1000;n=1,n<41,n++,money=(1+interest)money+deposit]
Print["At the end of ten years you will have $",money," in the bank"]
```

```
At the end of ten years you will have $6377.5 in the bank
```

In[113]:=
```
(* While statement *)
Clear[money,interest,deposit]
money=1000;deposit=100;interest=0.01;n=1;
While[n<41,money=(1+interest)money +deposit;n++]
Print["At the end of ten years you will have $",money," in the bank"]
```

```
At the end of ten years you will have $6377.5 in the bank
```