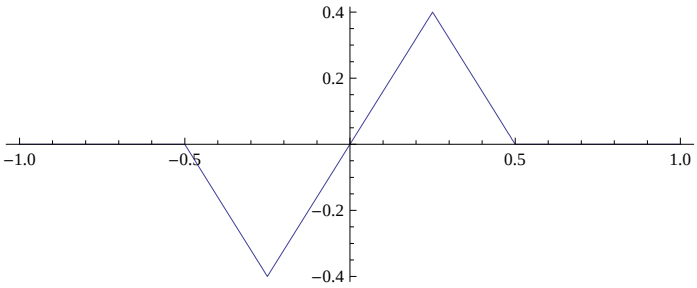
PHYS 301
HOMEWORK #4

Solutions

1. The function that is defined on (0, L) is :

$$f(x) = \begin{cases} 4\,h\,x/L, & 0 < x < L/4 \\ 2\,h - 4\,h\,x/L, & L/4 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

Since this string is not a repeating wave, we need to extend this form to the negative half plane so that we have a complete function that we can regard as 2 L periodic. We are told to expand the function in a sin series, so that means we want to make the odd extension between (0, -L). Our function will then become :



(where for purposes of specificity, I set L = 1 and h = 0.4). Since this is an odd function, we know a_0 and a_n will be zero, and we can compute b_n via :

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx =$$

$$\frac{2}{L}\left[\int_0^{L/4} 4\,h\,x/L\sin(n\pi x/L)\,dx + \int_{L/4}^{L/2}(2\,h - 4\,h\,x/L)\sin(n\pi x/L)\,dx\right]$$

computing via Mathematica :

```
Clear[L, h, x, b]
b[n_] := FullSimplify[
  (2 / L) (Integrate[4 h x / L Sin[n π x / L], {x, 0, L / 4}, Assumptions → n ∈ Integers] +
     Integrate[(2 h - 4 h x / L) Sin[n π x / L], {x, L / 4, L / 2}, Assumptions → n ∈ Integers])]
Do[Print["For n = ", n, "the Fourier sine coeffficient = ", b[n]], {n, 1, 12}]
```

For n = 1the Fourier sine coeffficient = $\frac{8\left(-1+\sqrt{2}\right)h}{\pi^2}$

For n = 2the Fourier sine coeffficient = $\frac{4h}{\pi^2}$

For n = 3the Fourier sine coeffficient = $\frac{8\left(1+\sqrt{2}\right)h}{9\pi^2}$

For n = 4the Fourier sine coeffficient = 0

For n = 5the Fourier sine coeffficient = $-\frac{8\left(1+\sqrt{2}\right)h}{25\pi^2}$

For n = 6the Fourier sine coeffficient = $-\frac{4h}{9\pi^2}$

For n = 7the Fourier sine coeffficient = $-\frac{8\left(-1+\sqrt{2}\right)h}{49\pi^2}$

For n = 8the Fourier sine coeffficient = 0

For n = 9the Fourier sine coeffficient = $\frac{8\left(-1+\sqrt{2}\right)h}{81\pi^2}$

For n = 10the Fourier sine coeffficient = $\frac{4h}{25\pi^2}$

For n = 11the Fourier sine coeffficient = $\frac{8\left(1+\sqrt{2}\right)h}{121\pi^2}$

For n = 12the Fourier sine coeffficient = 0

We integrate the function to determine the general expression for the coefficients :

```
FullSimplify[
 (2 / L) (Integrate[4 h x / L Sin[ n π x / L], {x, 0, L / 4}, Assumptions → n ∈ Integers] +
    Integrate[(2 h - 4 h x / L) Sin[n π x / L], {x, L / 4, L / 2}, Assumptions → n ∈ Integers])]
```

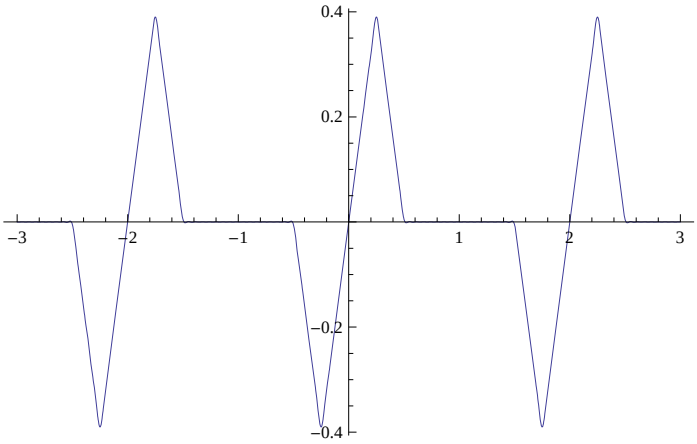
$$\frac{64\,h\,\text{Cos}\left[\frac{n\pi}{8}\right]\text{Sin}\left[\frac{n\pi}{8}\right]^3}{n^2\,\pi^2}$$

Our Fourier series is then :

$$f(x) = \frac{64\,h}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos[n\pi/8]\sin^3[n\pi/8]}{n^2}\,\text{Sin}[n\pi\,x/L]$$

and now setting h, L to specific values we verify via :

```
Clear[f, h, L, x]
h = 0.4; L = 1.0;
Plot[64 h / π^2 Sum[Cos[n π / 8] Sin[n π / 8]^3 Sin[n π x / L] / n^2, {n, 1, 33}], {x, -3 L, 3 L}]
```



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2. Consider the function described variations in atmospheric pressure :

$$p(t) = \begin{cases} 1, & 0 < t < 1/660 \\ 0, & 1/660 < t < 1/330 \\ -1, & 1/330 < t < 1/220 \end{cases}$$

Since the function is even, I include only the positive piece, knowing that symmetry arguments will allow me to integrate this function easily.

Now, if you look at the graph carefully, you should be able to see that there is as much area below the x axis as above, so we would expect our a_o coefficient to be zero. But let's verify:

$$a_o = \frac{1}{L}\int_{-1/220}^{1/220} p(t)\,dt = \frac{2}{L}\int_{0}^{1/220} p(t)\,dt = \frac{2}{1/220}\left[\int_0^{1/660} 1\cdot dt + \int_{1/330}^{1/220}(-1)\,dt\right] = 0$$

Since the function is 2 L periodic on (-1/220, 1/220), we use L = 1/220. We have an even function, so we already know the b terms are zero. We compute finally the a_n coefficients:

$$a_n = \frac{2}{(1/220)}\int_0^{1/220} p(t)\cos(220\,n\,\pi\,t)\,dt$$

$$= 440\left[\int_0^{1/660} 1\cdot\cos(220\,n\,\pi\,t)\,dt - \int_{1/330}^{1/220}\cos(220\,n\,\pi\,t)\,dt\right]$$

$$= 440\left[\frac{1}{220\,n\,\pi}\sin(220\,n\,\pi\,t)\Big|_0^{1/660} - \frac{1}{220\,n\,\pi}\sin(220\,n\,\pi\,t)\Big|_{1/330}^{1/220}\right]$$

$$= \frac{440}{220\,n\,\pi}[\sin(n\pi/3) - (\sin(n\pi) - \sin(2\,n\,\pi/3))]$$

$$= \frac{2}{n\,\pi}[\sin(n\pi/3) + \sin(2\,n\,\pi/3)]$$

These are the coefficients of the cos (220 n π) terms in the Fourier expansion. The first 8 coeffi-

cients are :

$$n = 1 \Rightarrow a_1 = \frac{4}{\pi} \sin(\pi/3) = \frac{2\sqrt{3}}{\pi}$$

$$a_2 = a_3 = a_4 = 0$$

$$a_5 = \frac{-2\sqrt{3}}{5\pi}$$

$$a_6 = a_8 = 0$$

$$a_7 = \frac{2\sqrt{3}}{7\pi}$$

Since intensities vary as the square of the amplitude, the relative intensities are (setting the n = 1 coefficient to a relative value of 1) :

$$1 : 0 : 0 : 0 : 1/25 : 0 : 1/49 : 0$$

We could write the Fourier series as :

$$p(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} (\sin(n\pi/3) + \sin(2n\pi/3)) \cos(220 n \pi t)/n$$

Verifying :

```
Plot[(2 / π) Sum[(Sin[n π / 3] + Sin[2 n π / 3]) Cos[220 n π t] / n, {n, 1, 71}],
 {t, -2 / 220, 2 / 220}]
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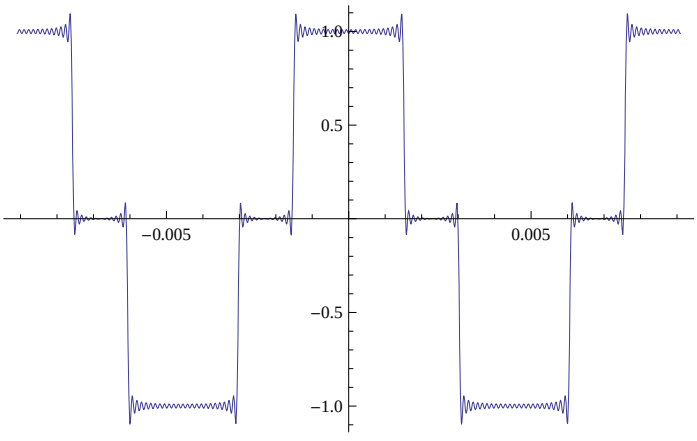

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3. The function shown in the graph for problem 10 - 2 is the 2 L periodic function with L = 1/262 s. You should be able to determine that the function is odd, so that we know :

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{L} \int_0^L p(t) \sin(n\pi t/L)\, dx$$

We can describe this piecewise function as :

$$p(t) = \begin{cases} 1, & 0 < t < 1/786 \\ 0, & 1/786 < t < 1/393 \\ 3, & 1/393 < t < 1/262 \end{cases}$$

We have then :

$$b_n = \frac{2}{(1/262)}\left[\int_0^{1/786} 1 \cdot \sin(262\, n\, \pi\, t)\, dt + \int_{1/393}^{1/262} 3 \cdot \sin(262\, n\, \pi\, t)\, dt\right]$$

$$= 2 \cdot 262\left[\frac{-1}{262\, n\, \pi}\cos(262\, n\, \pi\, t)\Big|_0^{1/786} - \frac{3}{262\, n\, \pi}\cos(262\, n\, \pi\, t)\Big|_{1/393}^{1/262}\right]$$

$$= \frac{-2}{n\pi}[(\cos(n\pi/3) - 1) + 3(\cos(n\pi) - \cos(2\, n\, \pi/3))]$$

You can reduce this expression further using trig identities, but we can work with this result. Let' s see what the first few coefficients look like :

$$b_1 = \frac{-2}{\pi}[(\cos(\pi/3) - 1) + 3((-1) - \cos(2\pi/3))] = \frac{-1}{\pi}\left[\left(\frac{1}{2} - 1\right) + 3\left(-1 - \left(\frac{-1}{2}\right)\right)\right] = \frac{4}{\pi}$$

$$b_2 = \frac{-2}{2\pi}[(\cos(2\pi/3) - 1) + 3(1 - \cos(4\pi/3))] = \frac{-1}{2\pi}[-1.5 + 4.5] = \frac{-3 \cdot 2}{2\pi}$$

$$b_3 = \frac{-2}{3\pi}[(\cos(\pi) - 1) + 3((-1) - \cos(2\pi))] = \frac{-1}{3\pi}[-8] = \frac{8 \cdot 2}{3\pi}$$

$$b_4 = \frac{-2}{4\pi}[(\cos(4\pi/3) - 1) + 3(1 - \cos(8\pi/3))] = \frac{-3 \cdot 2}{4\pi}$$

Continuing, we obtain :

$$b_5 = \frac{4}{5\pi},\ b_6 = 0,\ b_7 = \frac{4}{7\pi},\ b_8 = \frac{-4}{8\pi}$$

(You can compute an arbitrary list of coefficients via :)

```
Clear[b]
b[n_] := -2 / (n π) ((Cos[n π / 3] - 1) + 3 (Cos[n π] - Cos[2 n π / 3]))
Do[Print[b[n]], {n, 15}]
```

Intensity is related to the square of the amplitude; if we arbitrarily set the amplitude of b_1 =1, we have for relative intensities:

1 : 0.56 : 1.78 : 0.14 : 0.04 : 0 : 0.02 : 0.035

The Fourier series representing this pressure wave can be written as :

$$p(t) = \sum_{n=1}^{\infty} b_n \sin(262\, n\, \pi\, t)$$

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4. We are given a time varying current whose period is 1/60 s. The graph of this variation consists of half a sine wave from t = 0 to t = 1/120 s, and, is zero from 1/120 s to 1/60. In this case, 2 L =

1/60 so that L = 1/120 (the answer in the back is correct, although the statement about the value of L is misleading in the problem). The sine wave has amplitude of 5 amps and completes one half cycle in 1/120 s. If we recall that the equation for a wave can be written as :

$$g(t) = A \sin(2\pi ft)$$

where A is amplitude and f is frequency, we have that A = 5 amps and f = 60 Hz, so that

$$I(t) = \begin{cases} 5\sin(120\pi t), & 0 < t < 1/120 \\ 0, & 1/120 < t < 1/60 \end{cases}$$

We find the Fourier coefficients according to :

$$a_0 = \frac{1}{L}\int_0^{1/60} I(t)\,dt = 120\int_0^{1/120} 5\sin(120\pi t)\,dt = 600\left(\frac{-1}{120\pi}\cos(120\pi t)\right)\Big|_0^{1/120}$$

$$= \frac{-5}{\pi}(\cos\pi - 1) = \frac{10}{\pi}$$

$a_n =$

$$120\int_0^{1/120} 5\sin(120\pi t)\cos(120n\pi t)\,dt = \frac{5(1+\cos(n\pi))}{\pi(1-n^2)} = \begin{cases} 0, & n \text{ odd} \\ \frac{-10}{\pi(n^2-1)}, & n \text{ even} \end{cases}$$

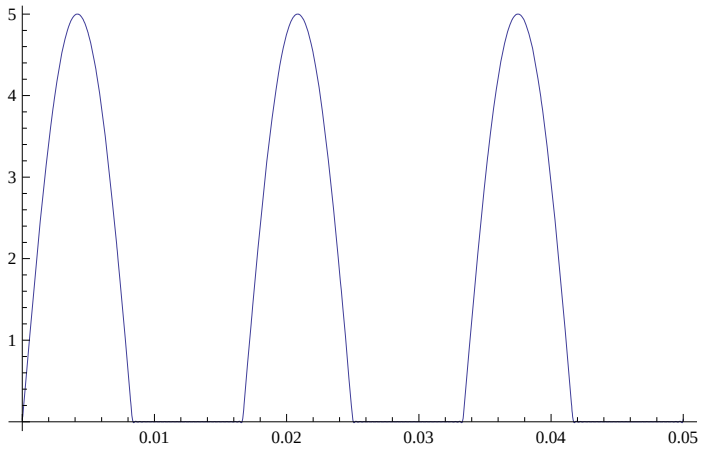
$$b_n = 120\int_0^{1/120} 5\sin(120\pi t)\sin(120n\pi t)\,dt = \begin{cases} 0, & n \neq 1 \\ \frac{5}{2}, & n = 1 \end{cases}$$

So the Fourier representation for this current becomes :

$$I(t) = \frac{5}{\pi} - \frac{10}{\pi}\sum_{n=2,\,\text{even}}^{\infty}\frac{\cos(120n\pi t)}{n^2-1} + \frac{5}{2}\sin(120\pi t)$$

I plot three full cycles of the function; note that one cycle consists of both a half sine wave and a portion where I (t) = 0. It is this complete cycle that is repeated every 1/60 s.

```
Clear[t]
Plot[
 5 / π - (10 / π) Sum[Cos[120 n π t] / (n^2 - 1), {n, 2, 52, 2}] + 2.5 Sin[120 π t], {t, 0, 3 / 60}]
```



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5. For our old friend,

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

we find the complex Fourier series using :

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{i n x}$$

where the c_n are found from:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i n x} dx$$

We find :

$$c_o = \frac{1}{2\pi} \int_{0}^{\pi} x \, dx = \frac{\pi}{4}$$

$$c_n = \frac{1}{2\pi} \int_{0}^{\pi} x e^{-i n x} dx = \frac{1}{2\pi}\left[\frac{-1}{i n} x e^{-i n x}\Big|_0^{\pi} - \left(\frac{-1}{i n}\right)\int_0^{\pi} e^{-i n x} dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{-1}{i n} x e^{-i n x}\Big|_0^{\pi} - \left(\frac{-1}{i n}\right)\left(\frac{-1}{i n}\right) e^{-i n x}\Big|_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-1}{i n} \pi e^{-i n \pi} - \frac{1}{i^2 n^2}\left(e^{-i n \pi} - 1\right)\right] = \frac{-1 (-1)^n}{2 i n} - \frac{1}{2\pi n^2}(1 - (-1)^n)$$

where we remember that $e^{-i n \pi} = \cos(n \pi) = (-1)^n$

Notice that our expression for the coefficients has a real part and an imaginary part. The real part will yield zero for even values of n, and gives

$$\text{Re}(c_n) = \frac{-2}{2\pi n^2} \text{ for n odd}$$

Therefore, we can write the first several terms of the complex Fourier expansion as :

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\left(\frac{e^{ix}}{2} + \frac{e^{-ix}}{2}\right) + \left(\frac{e^{3ix}}{2\cdot 3^2} + \frac{e^{-3ix}}{2\cdot(-3)^2}\right) + \left(\frac{e^{5ix}}{2\cdot 5^2} + \frac{e^{-5ix}}{2\cdot(-5)^2}\right) + \ldots\right) +$$
$$\left(\left(\frac{e^{ix}}{2i} - \frac{e^{-ix}}{2i}\right) + \left(-\frac{e^{-2ix}}{2\cdot 2i} + \frac{e^{-2ix}}{2\cdot 2i}\right) + \ldots\right)$$

Applying to the terms in parentheses the definitions of sin and cos :

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2} \quad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}$$

we can rewrite f (x) as :

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + \left(\sin x - \frac{\sin 2x}{2} + \ldots\right)$$

which matches the result for the trig series.