PHYS 301
HOMEWORK #5-- Solutions

1. For the first part of the question, we repeatedly contract Kronecker deltas :

$$\delta_{ij}\,\delta_{jk}\,\delta_{km}\,\delta_{im} \;=\; \delta_{ik}\,\delta_{km}\,\delta_{im} \;=\; \delta_{im}\,\delta_{im} \;=\; \delta_{im}\,\delta_{mi} = \delta_{mm} = 3$$

For the second, we recognize that we have three repeated indices, i, j, and k, so we are summing over all three indices. Further, we recall that the Levi - Civita permutation tensor is zero unless all three indices are different. Thus, we can write our expression as :

$$\epsilon_{ijk}\,\epsilon_{ijk} \;=\; \epsilon_{111}\,\epsilon_{111} + \epsilon_{112}\,\epsilon_{112} + \epsilon_{113}\,\epsilon_{113} + \ldots$$

and if we were to write out every term explicitly, we would have 27 terms on the right. However, we know that only six terms will be non - zero :

$$\epsilon_{ijk}\,\epsilon_{ijk} \;=\; \;=\; \epsilon_{123}\,\epsilon_{123} + \epsilon_{132}\,\epsilon_{132} + \epsilon_{213}\,\epsilon_{213} + \epsilon_{231}\,\epsilon_{231} + \epsilon_{312}\,\epsilon_{312} + \epsilon_{321}\,\epsilon_{321}$$

Each product on the right is either (1) (1) or (-1) (-1), so that the sum of all the terms is 6.

We can also make use of the ϵ - δ relationship. Expanding with respect to the subscript i :

$$\epsilon_{ijk}\,\epsilon_{ijk} \;=\; \delta_{jj}\,\delta_{kk} - \delta_{jk}\,\delta_{kj} \;=\; 3\cdot 3 \;-\; \delta_{jj} = \;3\cdot 3 - 3 \;=\; 6$$

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2. We translate our identity into Einstein summation notation :

$$\nabla\cdot(\mathrm{f}\,\mathbf{g}) \;=\; \frac{\partial}{\partial x_i}(\mathrm{f}\,g_i)$$

Remember, f is a scalar and has no components (so will not have any subscripts). Applying the product rule to this expression :

$$\frac{\partial}{\partial x_i}(\mathrm{f}\,g_i) = \;\mathrm{f}\,\frac{\partial}{\partial x_i}\,g_i \;+\; g_i\,\frac{\partial}{\partial x_i}\,\mathrm{f}$$

The first term on the right is the scalar f multiplied by the dot product between the del operator and g, or in other words : $\mathrm{f}\,\nabla\cdot\mathbf{g}$

The second term is the dot product of g with ∇f, so our identity is :

$$\nabla\cdot(\mathrm{f}\,\mathbf{g}) \;=\; \frac{\partial}{\partial x_i}(\mathrm{f}\,g_i) = \;\mathrm{f}\,\frac{\partial}{\partial x_i}\,g_i \;+\; g_i\,\frac{\partial}{\partial x_i}\,\mathrm{f} = \mathrm{f}\,\nabla\cdot\mathbf{g} + \mathbf{g}\cdot\nabla\mathrm{f}$$

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3. Using the identity from problem 2, we have :

$$\nabla\cdot\left(r^3\,\mathbf{r}\right) \;=\; r^3\,\nabla\cdot\mathbf{r} + \mathbf{r}\cdot\left(\nabla r^3\right) \tag{1}$$

The divergence of **r** is simply :

$$\nabla\cdot\mathbf{r} \;=\; \frac{\partial}{\partial x}\,x \;+\; \frac{\partial}{\partial y}\,y \;+\; \frac{\partial}{\partial z}\,z \;=\; 3$$

To find the gradient of the scalar r^3, we first write :

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

so that

$$r^3 = \left(x^2 + y^2 + z^2\right)^{3/2}$$

and :

$$\nabla r^3 = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{3/2} \hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{3/2} \hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{3/2} \hat{\mathbf{z}}$$

$$= \frac{3}{2}(2\,x)\sqrt{x^2 + y^2 + z^2}\;\hat{\mathbf{x}} + \frac{3}{2}(2\,y)\sqrt{x^2 + y^2 + z^2}\;\hat{\mathbf{y}} + \frac{3}{2}(2\,z)\sqrt{x^2 + y^2 + z^2}\;\hat{\mathbf{z}}$$

$$= 3\left(\sqrt{x^2 + y^2 + z^2}\right)\left(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}\right) = 3\,r\,\mathbf{r}$$

Substituting these results into the original equation (1) :

$$\nabla \cdot \left(r^3\, r\right) = 3\,r^3 + \mathbf{r} \cdot (3\,r\,\mathbf{r}) = 3\,r^3 + 3\,r\,r^2 = 6\,r^3$$

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4. Consider :

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A})$$

We know that **B×A** will produce a vector perpendicular to both **A** and **B**. Therefore, we expect a vanishing dot product for a vector **A** and a vector perpendicular to **A**. Let' s see if we can reproduce this result using summation notation. First, transform the expression to summation notation :

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) \rightarrow A_i\left(\epsilon_{ijk}\, B_j\, A_k\right) = B_j\, \epsilon_{ijk}\, A_k\, A_i \rightarrow \mathbf{B} \cdot (\mathbf{A} \times \mathbf{A})$$

At this point, you can successfully argue that any vector crossed with itself is zero since the angle between them is zero.