PHYS 301

HOMEWORK #6 -- Solutions

1. $\nabla \cdot (\mathbf{A}\times\mathbf{B}) \to \frac{\partial}{\partial x_i}\left(\epsilon_{ijk} A_j B_k\right)$

The term in parentheses produces the i^{th} component of **A×B;** the repeated index of "i" indicates we are taking the dot product of the del operator and the cross product. Now :

$$\frac{\partial}{\partial x_i}\left(\epsilon_{ijk} A_j B_k\right) = \epsilon_{ijk} \frac{\partial}{\partial x_i}\left(A_j B_k\right) = \epsilon_{ijk}\left(A_j \frac{\partial}{\partial x_i} B_k + B_k \frac{\partial}{\partial x_i} A_j\right) \quad \text{(via product rule)}$$

we distribute the ϵ :

$$= \epsilon_{ijk} A_j \frac{\partial}{\partial x_i} B_k + \epsilon_{ijk} B_k \frac{\partial}{\partial x_i} A_j = A_j \epsilon_{ijk} \frac{\partial}{\partial x_i} B_k + \epsilon_{ijk} B_k \frac{\partial}{\partial x_i} A_j$$

The first ϵ term on the right is the - j^{th} component of $\nabla\times\mathbf{B}$, and the second ϵ is the + kth component of $\nabla\times\mathbf{A}$. Thus we have :

$$= - A_j (\nabla \times B)_j + B_k (\nabla \times A)_k$$

summing over all components provides the final step in the proof :

$$= -\mathbf{A}\cdot(\nabla\times\mathbf{B}) + \mathbf{B}\cdot(\nabla\times\mathbf{A})$$

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2. $\nabla \times (\mathbf{A}\times\mathbf{B}) \to \epsilon_{mni} \frac{\partial}{\partial x_n} \epsilon_{ijk} A_j B_k = \epsilon_{imn} \epsilon_{ijk} \frac{\partial}{\partial x_n} A_j B_k$

(after permuting indices in ϵ_{mni} term)

then, apply ϵ - δ relationship :

$$\epsilon_{imn} \epsilon_{ijk} \frac{\partial}{\partial x_n} A_j B_k = \delta_{mj} \delta_{nk} \frac{\partial}{\partial x_n} A_j B_k - \delta_{mk} \delta_{nj} \frac{\partial}{\partial x_n} A_j B_k$$

In the first term, j = m and k = n; in the second term k = m and j = n leaving us with :

$$= \frac{\partial}{\partial x_n} A_m B_n - \frac{\partial}{\partial x_n} A_n B_m$$

Apply product rule to both partial derivatives :

$$= A_m \frac{\partial}{\partial x_n} B_n + B_n \frac{\partial}{\partial x_n} A_m - A_n \frac{\partial}{\partial x_n} B_m - B_m \frac{\partial}{\partial x_n} A_n$$

Recognizing that the product of terms with a repeated index indicates a dot product, we have :

$$= \mathbf{A}(\nabla\cdot\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} - \mathbf{B}(\nabla\cdot\mathbf{A})$$

3. $\nabla\times(f\,\mathbf{A}) \to \epsilon_{ijk}\frac{\partial}{\partial x_j}(f\,A_k) = \epsilon_{ijk}\,f\frac{\partial}{\partial x_j}A_k + \epsilon_{ijk}\,A_k\frac{\partial}{\partial x_j}f = f\,\epsilon_{ijk}\frac{\partial}{\partial x_j}A_k + \epsilon_{ijk}\,A_k\frac{\partial}{\partial x_j}f$

The first term all the way on the right is simply f times the + i^{th} component of curl **A**. The second term is the - i^{th} component of **A**×grad f. Summing over all components gives us :

$$\nabla\times(f\,\mathbf{A}) = f\,(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$$

4. Show $\nabla\times(\nabla\phi) = 0$

To show this, you must recall that you can interchange the order of differentiation in multivariable calculus, in other words :

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f\right)$$

If you compute this curl by hand, you should find that that the $\hat{x}$ component of the curl is the term :

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}\phi\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\phi\right) = 0$$

(A vector is zero iff all its components are zero, so we should expect to obtain similar terms for the $\hat{y}$ and $\hat{z}$ components.)

Now, using Einstein summation notation :

$$\nabla\times(\nabla\phi) \to \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}f\right)$$

Since the order of differentiation is interchangeable, we can write :

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}f\right) = \epsilon_{ijk}\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}f\right) \tag{1}$$

But the subscripts on the right are reversed, so we know that introduces a negative sign :

$$\epsilon_{ijk}\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}f\right) = -\,\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}f\right) \tag{2}$$

Equating eqs. (1) and (2) shows :

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}f\right) = -\,\epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}f\right)$$

If an expression equals its negative, the expression must be zero.

5. $\nabla(\phi\,\psi) \to \frac{\partial}{\partial x_i}(\phi\,\psi) = \phi\frac{\partial}{\partial x_i}\psi + \psi\frac{\partial}{\partial x_i}\phi = \phi\,(\nabla\psi) + \psi\,(\nabla\phi)$

Seems pretty easy at this point, doesn' t it?