

PHYS 301

HOMEWORK #7-- Solutions

1. The divergence theorem tells us :

$$\int \nabla \cdot \mathbf{r} \, d\tau = \int_S \mathbf{r} \cdot \mathbf{n} \, da$$

The divergence of the position vector is

$$\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

so that the volume integral is simply 3 times the volume of the sphere, so the value of the volume integral is

$$4 \pi R^3$$

To compute the surface integral, we first compute the value of $\mathbf{r} \cdot \mathbf{n}$. The dot product equals :

$$\mathbf{r} \cdot \mathbf{n} = |\mathbf{r}| |\mathbf{n}| \cos \theta$$

The magnitude of \mathbf{r} is R , and the magnitude of the unit normal is of course 1. On a sphere, the unit normal to the surface is perpendicular to the surface and lies along the radius vector, so the normal points in the same direction of \mathbf{r} . Thus, $\mathbf{r} \cdot \mathbf{n} = R$, and the surface integral becomes simply :

$$\int_S \mathbf{r} \cdot \mathbf{n} \, da = R \int_S da = R (4 \pi R^2) = 4 \pi R^3$$

and we have verified the divergence theorem for this case.

2. We know that the divergence theorem relates :

$$\int \nabla \cdot \mathbf{v} \, d\tau = \int_S \mathbf{v} \cdot \mathbf{n} \, da$$

We are given the surface integral over the surface of a cube. If we look at the integral, we recognize that the $dy \, dz$ term is integrated over the faces where the unit normal is in the \hat{x} direction. We can conclude that x^2 must be the \hat{x} component of the vector \mathbf{v} ; $dx \, dz$ is an element of area on the surfaces where the unit normal is in the \hat{y} direction, so that y^2 is the \hat{y} component of the vector. Similarly, z^2 must be the \hat{z} component of \mathbf{v} , therefore, we can deduce that:

$$\mathbf{v} = x^2 \hat{\mathbf{x}} + y^2 \hat{\mathbf{y}} + z^2 \hat{\mathbf{z}}$$

Using this vector, we determine :

$$\int \nabla \cdot \mathbf{v} \, d\tau = \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) \, dx \, dy \, dz = 3$$

In Mathematica, we can compute multiple integrals :

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In[2]:= Integrate[2 x + 2 y + 2 z, {x, 0, 1}, {y, 0, 1}, {z, 0, 1}]
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Out[2]= 3
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3. Stokes' theorem relates :

$$\int_S (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, da = \int_C \mathbf{v} \cdot d\mathbf{l}$$

First solving the integral on the left : Since the area is in the x - y plane, the normal is in the z direction. Therefore, we need only compute the z component of the curl, and the integral on the left becomes :

$$\int_S (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, da = \int_S (1 \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} \, da = \int_S 1 \, da = \text{area of square} = 1$$

To compute the complete line integral, we need to calculate four separate line integrals. Each line integral will be computed according to :

$$\int_C \mathbf{v} \cdot d\mathbf{l} = \int_C (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}) \cdot (dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}) = \int_C (v_x dx + v_y dy + v_z dz)$$

1) Along the x axis from (0, 0) to (1, 0), $dy = dz = 0$ and $z = 0$, so the value of this line integral is zero.

2) Along the line from (1, 0) to (1, 1), $dx = dz = 0$, and $x = 1$, so the line integral along this path is 1.

3) Along the line from (1, 1) to (0, 1), $dy = dz = 0$ and $z = 0$, so the line integral is zero.

4) From (0, 1) to (0, 0), $dx = dz = 0$ and $x = 0$, so the line integral is zero.

Summing these four contributions, the line integral around the entire contour is 1, verifying Stokes' Theorem.

4. Proceeding as in the previous problem, we compute the $\hat{\mathbf{z}}$ component of the curl and find that :

$$\int_S (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, da = \int_S 2x \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} \, da = \int_0^1 \int_0^1 2x \, dx \, dy = 1$$

Computing the line integrals along each edge of the square :

1) From (0, 0) to (1, 1): $dy = dz = 0$ and $z = 0 \Rightarrow \int z^2 dx = 0$

2) From (1, 0) to (1, 1), $dx = dz = 0$ and $x = 1 \Rightarrow \int_0^1 x^2 dy = 1$

3) From (1, 1) to (0, 1), $dy = dz = 0$ and $z = 0$, so $\int_1^0 z^2 dx = 0$

4) From (0, 1) to (0, 0), $dx = dz = 0$ and $x = 0$; $\int_1^0 x^2 dy = 0$

5. In this case, our area is in the plane $z = -3$; the normal to this plane is . Computing the $\hat{\mathbf{z}}$ component of the curl yields :

$$\int_S (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, da = \int_S (z^3 - 1) \, da$$

For this area, $z = -3$, so the integral is simply $\int (-27 - 1) \, da = -28 (4\pi) = -112\pi$

The contour is the circle of radius 2 in the plane $z = -3$, so the area of the circle is 4π .

To compute the line integral, we parameterize our integral according to :

$$\begin{aligned} x &= 2 \cos \theta & dx &= -2 \sin \theta \, d\theta & y &= 2 \sin \theta & dy &= 2 \cos \theta \, d\theta \\ & & & & & & \text{and } z &= -3 \text{ and } dz = 0 \end{aligned}$$

Therefore, the line integral becomes :

$$\begin{aligned} \int_C \mathbf{v} \cdot d\mathbf{l} &= \int_C (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}) \cdot (dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}) = \int_C (v_x dx + v_y dy + v_z dz) \\ &= \int_0^{2\pi} \left(2 \sin \theta (-2 \sin \theta \, d\theta) + 2 \cos \theta (-3)^3 (2 \cos \theta \, d\theta) \right) = \int_0^{2\pi} (-4 \sin^2 \theta \, d\theta - 108 \cos^2 \theta \, d\theta) \\ &= -112\pi \end{aligned}$$

since $\int_0^{2\pi} \sin^2 \theta \, d\theta = \int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$