PHYS 301
HOMEWORK #9-- Solutions

1. For this problem, it is important to remember that if a vector field is conservative (i.e., its curl = 0), then the vector can be derived from a scalar potential. Additionally, if the field is conservative, its line integral is path independent, and can be computed either directly as a line integral, or by finding the value of its potential at the end points. For the two vectors given, a) is non - conservative (it should be easy to show its curl is non - zero) and b) is conservative.

Therefore, we have to compute the line integral for case a) explicitly for the two paths given. Recall that :

$$\int_C \mathbf{F} \cdot d\mathbf{l} = \int_C \left(F_x\, dx + F_y\, dy + F_z\, dz\right)$$

For the vector in case a) :

$$F_x = x\,y \quad F_y = 2\,y\,z \qquad F_z = 3\,x\,z$$

so the line integral becomes :

$$\int_C (x\,y\,dx + (2\,y\,z\,dy + 3\,x\,z\,dz)$$

The first path is the straight line y = x, so we can parameterize this integral as x = y = t; dx = dy = dt. Since we are in the xy plane, z = dz = 0, and the line integral is simply :

$$\int_0^1 t^2\, dt = \frac{1}{3}$$

Along the path $y = x^2$, we set:

$$x = t;\ dx = dt;\ y = t^2;\ dy = 2\,t\,dt'\,z = dz = 0$$

and the line integral becomes :

$$\int_0^1 t^3\, dt = \frac{1}{4}$$

The second vector field is conservative, so we should expect the line integral to be the same for both paths (but let' s do both to verify). Using the parameterization x = y = t; dx = dy = dt; z = dz = 0 :

$$\int_0^1 \left(t^2\, dt + 2\,t^2\, dt\right) = 1$$

Along the curved path, we set :

$$x = t \quad dx = dt \quad y = t^2 \quad dy = 2\,t\,dt \quad z = dz = 0$$

and the line integral becomes :

$$\int_0^1 \left(t^4\,dt + 2\,t^3\,(2\,t\,dt)\right) = \int_0^1 5\,t^4\,dt = 1$$

as we expect.

We can also find the potential from which this force is derived and find the work by evaluating the potential at the end points of the path. We can write :

$$\mathbf{F}_2 = \nabla\phi \Rightarrow \left\{y^2, 2\,x\,y + z^2, 2\,y\,z\right\} = \left\{\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right\}$$

Equating the x components of each vector and integrating gives us :

$$\frac{\partial\phi}{\partial x} = y^2 \Rightarrow \phi = y^2\,x + g\,(y, z)$$

where the constant of integration is constant with respect to x, so to be as general as we can be we write the constant as a function of y and z. If we differentiate this expression for ϕ with respect to y, and equate it to the y component of the vector we obtain :

$$\frac{\partial\phi}{\partial y} = 2\,y\,x + g'\,(y, z) = 2\,x\,y + z^2$$

This means that g' (y, z) = z^2 . Integrating this tells us g(y,z) = z^2y + h (z). Substituting this back into our expression for ϕ gives:

$$\phi = x\,y^2 + y\,z^2 + h\,(z)$$

Now, we differentiate ϕ with respect to z and equate with the z component of the vector :

$$\frac{\partial\phi}{\partial z} = 2\,y\,z + h'\,(z) = 2\,y\,z \Rightarrow h'\,(z) = 0 \text{ and } h\,(z) = \text{numerical constant}$$

Thus, apart from a numerical constant, we know that

$$\phi = x\,y^2 + y\,z^2$$

Finally, we compute the work from :

$$W = \phi\,(1, 1, 0) - \phi\,(0, 0, 0) = 1$$

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2. To find the scale factors, we first find expressions for dx, dy and dz :

$$dx = v\cos\theta\,du + u\cos\theta\,dv - u\,v\sin\theta\,d\theta$$
$$dy = v\sin\theta\,du + u\sin\theta\,dv + u\,v\cos\theta\,d\theta$$
$$dz = u\,du - v\,dv$$

Well, we can write out lots of terms and add, or we can be appropriately lazy :

In[249]:=
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Clear[dx, dy, dz, du, dv, dθ, u, v, θ]
dx = v Cos[θ] du + u Cos[θ] dv - u v Sin[θ] dθ;
dy = v Sin[θ] du + u Sin[θ] dv + u v Cos[θ] dθ;
dz = u du - v dv;
Expand[dx^2 + dy^2 + dz^2]
```

Out[253]= $du^2\,u^2 - 2\,du\,dv\,u\,v + dv^2\,v^2 + dv^2\,u^2\,\text{Cos}[\theta]^2 + 2\,du\,dv\,u\,v\,\text{Cos}[\theta]^2 + du^2\,v^2\,\text{Cos}[\theta]^2 + d\theta^2\,u^2\,v^2\,\text{Cos}[\theta]^2 + dv^2\,u^2\,\text{Sin}[\theta]^2 + 2\,du\,dv\,u\,v\,\text{Sin}[\theta]^2 + du^2\,v^2\,\text{Sin}[\theta]^2 + d\theta^2\,u^2\,v^2\,\text{Sin}[\theta]^2$

That' s ugly; let' s simplify :

In[254]:=
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Simplify[%]
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Out[254]= $d\theta^2\, u^2\, v^2 + du^2\,(u^2+v^2) + dv^2\,(u^2+v^2)$

Ok, no cross terms, it' s an orthogonal transformation, and the scale factors are :

$$h_u = h_v = \sqrt{u^2+v^2} \qquad h_\phi = u\,v$$

To find unit vectors, write the position vector

$$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} = u\,v\cos\theta\,\hat{\mathbf{x}} + u\,v\sin\theta\,\hat{\mathbf{y}} + \frac{1}{2}\left(u^2-v^2\right)\hat{\mathbf{z}}$$

and each unit vector is determined from :

$$\hat{\mathbf{q}}_{\mathbf{i}} = \frac{\partial\mathbf{r}}{\partial q_i} \Big/ \left|\frac{\partial\mathbf{r}}{\partial q_i}\right|$$

where $\hat{\boldsymbol{q}}_i$ is the generalized unit vector. Applying this definition:

$$\hat{\mathbf{u}} = \frac{v\cos\theta\,\hat{\mathbf{x}} + v\sin\theta\,\hat{\mathbf{y}} + u\,\hat{\mathbf{z}}}{\sqrt{u^2+v^2}}$$

$$\hat{\mathbf{v}} = \frac{u\cos\theta\,\hat{\mathbf{x}} + u\sin\theta\,\hat{\mathbf{y}} - v\,\hat{\mathbf{z}}}{\sqrt{u^2+v^2}}$$

$$\hat{\boldsymbol{\theta}} = \frac{-u\,v\sin\theta\,\hat{\mathbf{x}} + u\,v\cos\theta\,\hat{\mathbf{y}}}{u\,v} = -\sin\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}$$

And it is easy to take dot products of the vectors to show that the basis vectors are orthogonal.

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3. See solution in separate link.

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4. We make the substitutions :

$$\nu = \frac{c}{\lambda} \text{ and } d\,\nu = \frac{-c}{\lambda^2}\,d\lambda$$

to obtain :

$$B_\lambda(T)\,d\lambda = 2\,h\,\frac{\left(\frac{c}{\lambda}\right)^3}{c^2}\,\frac{1}{e^{h\,c/\lambda\,k\,T}-1}\,\frac{c}{\lambda^2}\,d\lambda = 2\,h\,\frac{c^2}{\lambda^5}\left(e^{h\,c/\lambda\,k\,T}-1\right)^{-1}d\lambda$$

Using Mathematica :

In[255]:=
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Clear[b, λ, t, h, c, k]
b[λ_, t_] := b[λ, t] = (2 h c^2 / λ^5) (Exp[h c / (λ k t)] - 1)^-1
D[b[λ, t], λ]
```

Out[257]= $$\frac{2\,c^3\,e^{\frac{c\,h}{k\,t\,\lambda}}\,h^2}{\left(-1+e^{\frac{c\,h}{k\,t\,\lambda}}\right)^2 k\,t\,\lambda^7} - \frac{10\,c^2\,h}{\left(-1+e^{\frac{c\,h}{k\,t\,\lambda}}\right)\lambda^6}$$

If you set this equal to zero and cancel common factors, you obtain :

$$\frac{\frac{h\,c}{\lambda\,k\,t}\,e^{h\,c/\lambda\,k\,t}}{e^{h\,c/\lambda\,k\,t}-1}=5$$

If we set x = h c/λ k t, this equation becomes :

$$\frac{x\,e^{x}}{e^{x}-1}=5$$

Which can be solved either via :

In[259]:=
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Solve[x Exp[x] == 5 (Exp[x] - 1), x] // N
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Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[259]=
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{{x → 0.}, {x → 4.96511}}
```

or via :

In[261]:=
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FindRoot[x Exp[x] == 5 (Exp[x] - 1), {x, 5}]
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Out[261]=
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{x → 4.96511}
```

Using standard MKS values for h, c and k and this value of x allows us to write Wien' s Law :

$$x = \frac{hc}{\lambda\,k\,T} \Rightarrow \lambda_{max} = \frac{hc}{x\,k\,T} = \frac{0.0029}{T}$$

where T is measured in Kelvins and λ in meters. The sun' s radiating temperature is approximately 5800 K, so the wavelength of maximum energy emission is 500 nm ($5\ 10^{-7}$m)