## PHYS 301 <br> HOMEWORK \#10-- Solutions

1. There are several elements to this question. First, this question revolves around the equation of continuity, motivated in class and described in the text :

$$
\frac{\partial \rho}{\partial \mathrm{t}}+\nabla \cdot=0
$$

Now, we have to be very careful in this problem not to confuse the meaning of the symbol $\rho$. The $\rho$ used in the partial time derivative refers to the density of water. Being told that water is incompressible means that $\partial \rho / \partial \mathrm{t}=0$, and the continuity equation becomes simply :

$$
\nabla \cdot=0 .
$$

where $v$ is the velocity vector. Because the water is flowing outward from the origin, the velocity is a function only of the radial direction. Second, the geometry of the situation suggests the use of plane polar coordinates, so we can write that $\mathrm{v}=\mathrm{v}(\rho)$ and now $\rho$ refers to the distance from the origin.

In polar coordinates, the divergence is:

$$
\nabla \cdot \mathbf{v}=\frac{1}{\rho}\left[\left(\frac{\partial}{\partial \rho}\left(\rho \mathrm{v}_{\rho}\right)+\frac{\partial}{\partial \phi}\left(\mathrm{v}_{\phi}\right)+\frac{\partial}{\partial \mathrm{z}}\left(\rho \mathrm{v}_{\mathrm{z}}\right)\right)\right]
$$

where $v_{\rho}$ is the velocity in the $\hat{\rho}$ (radial) directionIn this case, since the only component of velocity is radial, we have

$$
=\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \mathrm{v}_{\rho}\right)=0
$$

This is a particularly simple expression. Since the derivative of $\rho v_{\rho}$ is zero, we know that $\rho v_{\rho}$ is a constant, so we can write:

$$
\mathrm{v}_{\rho}=\frac{\mathrm{k}}{\rho}
$$

where k is some constant.
2. In this problem we will use our knowledge of vector calculus to transform the integral form of ampere' s law to the differential form. We start with :

$$
\oint \mathbf{B} \cdot \mathrm{d} \mathbf{l}=\mu_{\mathrm{o}} \mathrm{I}
$$

where I is the current enclosed in the loop. Stokes' theorem allows us to relate a line integral to a surface integral, so we get:

$$
\begin{equation*}
\oint \mathbf{B} \cdot \mathrm{d} \mathbf{l}=\int_{\mathrm{S}}(\nabla \times \mathbf{B}) \cdot \mathrm{d} \mathbf{a}=\mu_{\mathrm{o}} \mathrm{I} \tag{1}
\end{equation*}
$$

where da is an element of the area of the surface defined by the loop above. We are told that

$$
\mathrm{I}=\int_{\mathrm{S}} \mathbf{J} \cdot \mathrm{~d} \mathbf{a}
$$

using this definition of I in equation (1) gives us:

$$
\int_{\mathrm{s}}(\nabla \times \mathbf{B}) \cdot \mathrm{d} \mathbf{a}=\mu_{\mathrm{o}} \int_{\mathrm{S}} \mathbf{J} \cdot \mathrm{~d} \mathbf{a}
$$

If the two integrals are the same over any surface, the integrands must be the same, and thus:

$$
\nabla \times \mathbf{B}=\mu_{\mathrm{o}} \mathbf{J}
$$

3. This is a problem in which we will investigate some of the nuances of vector field theory. You encounter conservative forces in many courses, and are taught that conservative forces have several properties, including:

- the line integral of a conservative force around a closed loop is zero
- a conservative force has a curl of zero
- a conservative force can be derived from a scalar potential

Usually, if a vector field has any one of these properties, it all of these properties. In other words, if you can show a vector field has a zero curl, then you can (usually assume) that it is conservative, that it's line integral on a closed curve will be zero, and that the vector is the gradient of some scalar. Usually.

This problem highlights some of the subtleties involved in understanding conservative forces and vector fields. In the first part of the problem, you show that the line integral around a closed loop is not zero, hence we know absolutely definitely that the field is not conservative. Yet, in part b we find that the field has a zero curl (also known as irrotational). Finally, in part c we show that there is a scalar function $S$ whose gradient equals the vector field $\mathbf{B}$. We are asked in parts $b$ ) and $c$ ) to explain why the field is not conservative (which we know definitely from part a) even though it exhibits properties of conservative fields.
a) We will take the line integral of B along a circle of radius $\rho$ (remember we have to write both the vector $\mathbf{B}$ and the element of length dl in terms of polar coordinates) :

$$
\oint \mathbf{B} \cdot \mathrm{d} \mathbf{l}=\int\left(\mu_{\mathrm{o}} \frac{\mathrm{I}}{2 \pi \rho} \hat{\boldsymbol{\phi}}\right) \cdot \mathrm{d} \mathbf{l}=\int\left(\mu_{\mathrm{o}} \frac{\mathrm{I}}{2 \pi \rho} \hat{\boldsymbol{\phi}}\right) \cdot(\rho \mathrm{d} \phi \hat{\boldsymbol{\phi}})
$$

(The element of length can be expressed as :

$$
\mathrm{d} \mathbf{l}=\mathrm{h}_{\mathrm{i}} \mathrm{dq}_{\mathrm{i}} \hat{\mathbf{q}}_{\mathbf{i}}=\mathrm{d} \rho \hat{\boldsymbol{\rho}}+\rho \mathrm{d} \phi \hat{\boldsymbol{\phi}}+\mathrm{dz} \hat{\mathbf{z}}
$$

Since $\mathbf{B}$ has only a $\phi$ component, we need only consider the $\phi$ component of $\mathrm{d} \mathbf{l}$ in the dot product.
This gives us a very simple integral :

$$
\int_{0}^{2 \pi} \frac{\mu_{\mathrm{o}} \mathrm{I}}{2 \pi \rho}(\rho \mathrm{~d} \phi)=\mu_{\mathrm{o}} \mathrm{I}
$$

Since the line integral around a closed loop is non-zero, the force is non conservative.
b) A vector is irrotational if its curl is zero. It is easy to compute the curl of $B$ and show it is zero . This does not imply a conservative field because the region is not simply connected; the wire itself creates a hole in the region, so it does not meet the criteria for a conservative field.
c) The first part of this problem is easy. If you define a scalar function S such that :

$$
\mathrm{S}=\frac{\mu_{\mathrm{o}} \mathrm{I}}{2 \pi} \phi
$$

it is easy to compute the gradient. The gradient in cylindrical coordinates is:

$$
\nabla \mathrm{S}=\frac{\partial \mathrm{S}}{\partial \rho} \hat{\rho}+\frac{1}{\rho} \frac{\partial \mathrm{~S}}{\partial \phi} \hat{\boldsymbol{\phi}}+\frac{\partial \mathrm{S}}{\partial \mathrm{z}} \hat{\mathbf{z}}
$$

Since $S$ depends only on $\phi$, the gradient becomes

$$
\nabla \mathrm{S}=\frac{1}{\rho} \frac{\mu_{\mathrm{o}} \mathrm{I}}{2 \pi} \hat{\boldsymbol{\phi}}
$$

which is equivalent to the original vector $\mathbf{B}$. So why can't we say that $\mathbf{B}$ is derived from the gradient of a scalar potential, S? This part is subtle. Look at the function S, and imagine its value just above and below the +x axis. Remember that $\phi$, the azimuthal angle, is the only variable upon which the value of $S$ depends. If a point is just above the $x$ axis, then the value of $S$ at that point will be close to zero, since $\phi$ is infinitesimally above zero. Now, consider a particle just below the +x axis; the value of S is nearly $\mu \mathrm{I}$ since the value of $\phi$ approaches $2 \pi$. In the language you learned in Calc I, the limit of S from above approaches zero, while the limit of S from below approaches $\mu_{o} \mathrm{I}$ (since $\phi$ approaches $2 \pi$ ). Since the limit from above is not equal to the limit from below, this means $S$ is not differentiable everywhere, and therefore does not have a well defined gradient.
4. The divergence in spherical coordinates of a vector V is :

$$
\nabla \cdot \mathbf{V}=\frac{1}{\mathrm{r}^{2}} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^{2} \mathrm{~V}_{\mathrm{r}}\right)+\frac{1}{\mathrm{r} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \mathrm{~V}_{\theta}\right)+\frac{1}{\mathrm{r} \sin \theta} \frac{\partial \mathrm{~A}_{\phi}}{\partial \phi}
$$

In spherical coordinates, the position vector (as you determined in an earlier homework) is:

$$
\mathbf{r}=\mathrm{r} \hat{\mathbf{r}}
$$

and so has only a radial component. Then, $\nabla \cdot \mathrm{r}$ is:

$$
\nabla \cdot \mathbf{r}=\frac{1}{\mathrm{r}^{2}} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^{2} \cdot \mathrm{r}\right)=\frac{3 \mathrm{r}^{2}}{\mathrm{r}^{2}}=3 \text { as it must. }
$$

5. We are asked to find the values of $n$ which will satisfy Laplace's equation if the scalar potential is :

$$
\mathrm{V}=\mathrm{cr}^{\mathrm{n}}
$$

Since the potential has only a radial dependence, we only need the radial part of the Laplacian:

$$
\nabla^{2} \mathrm{~V}=\frac{1}{\mathrm{r}^{2}} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^{2} \frac{\partial}{\partial \mathrm{r}} \mathrm{~V}_{\mathrm{r}}\right)=0
$$

Substitute the expression for V :

$$
\begin{gathered}
\nabla^{2} \mathrm{~V}=\frac{1}{\mathrm{r}^{2}} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^{2} \frac{\partial}{\partial \mathrm{r}} \mathrm{~V}_{\mathrm{r}}\right)=\frac{1}{\mathrm{r}^{2}} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^{2} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{cr}^{\mathrm{n}}\right)\right)= \\
\frac{1}{\mathrm{r}^{2}} \frac{\partial}{\partial \mathrm{r}}\left(\mathrm{crn} \mathrm{r}^{\mathrm{n}+1}\right)=\frac{1}{\mathrm{r}^{2}} \mathrm{c} \mathrm{n}(\mathrm{n}+1) \mathrm{r}^{\mathrm{n}-2}=0 \Rightarrow \mathrm{n}=0,-1
\end{gathered}
$$

6. Here we are asked to combine several summations into one. We have to make sure each sum has the same lower and upper limits, and each has $x$ raised to the same exponent. For simplicity, I will not type in explicitly the upper limits on the sums. We should recognize that the upper limit is always $\infty$.

We are given:

$$
\sum_{\mathrm{n}=0} \mathrm{n}(\mathrm{n}-1) \mathrm{x}^{\mathrm{n}-2}+\sum_{\mathrm{n}=0} \mathrm{n} \mathrm{x}^{\mathrm{n}-1}+\sum_{\mathrm{n}=0} \mathrm{x}^{\mathrm{n}}
$$

Let' s follow the practice shown in class and re - index each summation so that the exponent is $n$. In the first sum, set $\mathrm{k}=\mathrm{n}-2$ ( or $\mathrm{n}=\mathrm{k}+2$ ), in the middle sum set $\mathrm{k}=\mathrm{n}-1$, and we do not have to reindex the final sum. Making these changes wherever $n$ appears (including in the lower summation limits), we get:

$$
\sum_{n=-2}^{\infty}(n+2)(n+1) x^{n}+\sum_{n=-1}^{\square}(n+1) x^{n}+\sum_{n=0}^{\square} x^{n}
$$

Now we have to get all the sums to have the same lower limit, we will do this by stripping out the $n$ $=-2$ and $n=-1$ terms from the first sum and the $n=-1$ term from the second sum. This gives us:

$$
\begin{aligned}
& \quad(-2+2)(-2+1) x^{-2}+(-1+2)(-1+1) x^{-2}+ \\
& (-1+1) x^{-1}+\sum_{n=0}(n+2)(n+1) x^{n}+\sum_{n=0}(n+1) x^{n}+\sum_{n=0} x^{n} \\
& =0 x^{-2}+0 x^{-1}+0 x^{-1}+\sum_{n=0}[(n+2)(n+1)+(n+1)+1] x^{n}
\end{aligned}
$$

$$
\begin{gathered}
=\sum_{n=0}\left[n^{2}+3 n+2+n+1+1\right] x^{n}=\sum_{n=0}\left[n^{2}+4 n+4\right] x^{n} \\
\text { or } \sum_{n=0}(n+2)^{2} x^{n}
\end{gathered}
$$

