PHYS 301
HOMEWORK #11-- Solutions

1. Start with :

$$\left(1+x^2\right) y'' - 2\,x\,y' + 2\,y = 0$$

Assume a solution of the form :

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and we obtain:

$$\left(1+x^2\right) \sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} - 2 \sum_{n=1}^{\infty} n\,a_n\,x^n + 2 \sum_{n=0}^{\infty} a_n\,x^n = 0$$

Distribute the terms in parens:

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} + \sum_{n=2}^{\infty} n\,(n-1)\,x^n - 2 \sum_{n=1}^{\infty} n\,a_n\,x^n + 2 \sum_{n=0}^{\infty} a_n\,x^n = 0$$

Re - index the first term by setting k = n - 2 (and after making all changes, set n = k since the choice of variable is irrelevant)

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,x^n + \sum_{n=2}^{\infty} n\,(n-1)\,x^n - 2 \sum_{n=1}^{\infty} n\,a_n\,x^n + 2 \sum_{n=0}^{\infty} a_n\,x^n = 0$$

"Strip out" terms so that all sums have the same lower limit of n = 2 :

$$2\,a_2 + 6\,a_3\,x - 2\,a_1\,x + 2\,a_0 + 2\,a_1\,x + \sum_{n=2}^{\infty} [(n+2)\,(n+1)\,a_{n+2} + (n\,(n-1) - 2\,n + 2)\,a_n]\,x^n = 0$$

The terms inside the summation tell us that either x is identically zero (the trivial case) , or that :

$$a_{n+2} = \frac{2\,n - n\,(n+1) - 2}{(n+2)\,(n+1)}\,a_n = -\frac{(n-2)\,(n-1)\,a_n}{(n+2)\,(n+1)}$$

The terms in the numerator will go to zero when n = 2 or n = 1; this means that $a_3 = a_4 = 0$. Since all higher coefficients are linked to either a_3 or a_4, we know all coefficients higher than a_2 will be zero. At most then, there are three non zero terms in this solution. We use the “stripped out” terms to find the values of the first few coefficients. Combining these terms, we get:

$$2\,a_2 + 2\,a_0 = 0 \qquad \text{and} \qquad 6\,a_3\,x = 0$$

The first relationship tells us that $a_2 = -a_o$, and the second tells us that $a_3 = 0$ (which we already knew from the recursion relation). Notice that the stripped out terms give us no information about a_1. This means that a_1 is a free variable that can only be determined with boundary conditions. Thus, our series solution is:

$$y = a_0 + a_1 x + a_2 x^2 = a_o\left(1 - x^2\right) + a_1 x$$

To show that this solution satisfies the original differential equation:

In[58]:=
```
Clear[a0, a1, x, f]
f[x_] := a0 (1 - x^2) + a1 x
Simplify[(1 + x^2) f''[x] - 2 x f'[x] + 2 f[x]]
```

Out[60]= 0

And we see that this solution satisfies the differential equation.

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2. Start with :

$$2 y' + 3 y = \sqrt{x + 1} \qquad y(0) = 1$$

Our trial solution is:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_o + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

and :

$$y' = a_1 + 2 a_2 x + 3 a_3 x^2 + 4 a_4 x^3 + 5 a_5 x^4$$

Multiplying by the proper coefficients and adding, the LHS becomes:

$$(3 a_0 + 2 a_1) + (3 a_1 + 4 a_2) x + (3 a_2 + 6 a_3) x^2 + (3 a_3 + 8 a_4) x^3 + (3 a_4 + 10 a_5) x^4$$

And these must equal the analogous coefficients on the right. We expand $\sqrt{1+x}$ in a Maclaurin series around x_0=0:

$$f(0) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} \Rightarrow f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{-1}{4}(x+1)^{-3/2} \Rightarrow f''(0) = \frac{-1}{4}$$

$$f'''(x) = \frac{3}{8}(x+1)^{-5/2} \Rightarrow f'''(0) = \frac{3}{8}$$

(I will leave the calculation of the next two derivatives to you, you should find that the fourth derivative evaluated at zero = -15/16, and the fifth is + 105/32)

The series expansion is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)\, x^n}{n!} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5 x^4}{128} + \frac{7 x^5}{256}$$

Remember, all the terms on the left must equal all the terms on the right, and we are told that y (0) = 1. If you substitute x = 0 into the original power series expansion, we get that a_0 = 1. With this information, we can compute the entire power series:

$(3\,a_o + 2\,a_1) = 1$ (since the x^0 coefficients must match); or $3 + 2\,a_1 = 1 \Rightarrow a_1 = -1$

$$3\,a_1 + 4\,a_2 = \frac{1}{2} \Rightarrow -3 + 4\,a_2 = \frac{1}{2} \Rightarrow a_2 = \frac{7}{8}$$

$$3\,a_2 + 6\,a_3 = \frac{-1}{8} \Rightarrow \frac{21}{8} + 6\,a_3 = \frac{-1}{8} \Rightarrow a_3 = -\frac{22}{48} = \frac{-11}{24}$$

$$3\,a_3 + 8\,a_4 = \frac{1}{16} \Rightarrow \frac{-33}{24} + 8\,a_4 = \frac{1}{16} \Rightarrow a_4 = \frac{23}{128}$$

$$3\,a_4 + 10\,a_5 = \frac{-5}{128} \Rightarrow \frac{69}{128} + 10\,a_5 = \frac{-5}{128} \Rightarrow a_5 = \frac{-37}{640}$$

so our power series is:

$$y = 1 - x + \frac{7}{8}x^2 - \frac{11}{24}x^3 + \frac{23}{128}x^4 - \frac{37}{640}x^5$$

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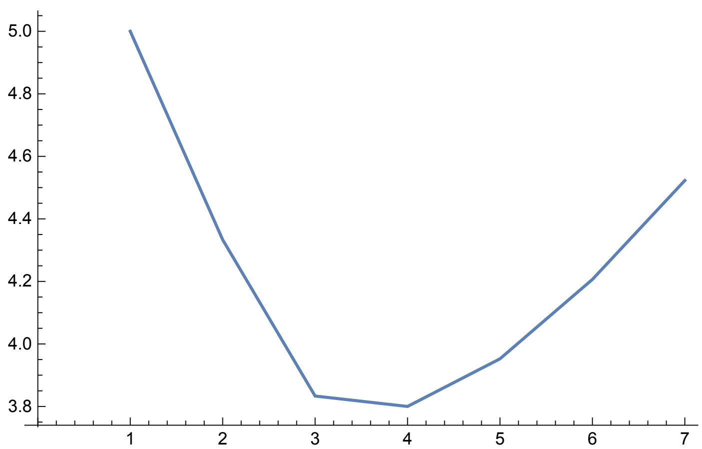
3. A program could look like this :

In[78]:=
```
Clear[f, x, n, integral]
f[x_, n_] := x^n
integral[n_] := Integrate[D[f[x, n], x]^2 + 10 x f[x, n], {x, 0, 1}]
ListLinePlot[Table[integral[n], {n, 0, 6}]]
```

Out[81]=



Which shows you that n =4 is the integer value that minimizes the value of the function on [0,1].

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4. If we adopt a system in which down is positive, the force of gravity will act down and the frictional force will act up, hence Newton' s second law becomes :

$$\Sigma F = m\frac{dv}{dt} = m\,g - c\,v$$

We can solve this equation simply via separation of variables:

$$\frac{m}{m\,g - c\,v}dv = dt$$

$$\frac{-m}{c} \ln (m g - c v) = t + K \text{ (where K is a constant)}$$

standard algebra leads to:

$$v = \frac{1}{c}\left(m g - A e^{-c t/m}\right)$$

setting v (0) = 0 gives us that A = m g and we can write:

$$v = \frac{m g}{c}\left(1 - e^{-c t/m}\right)$$

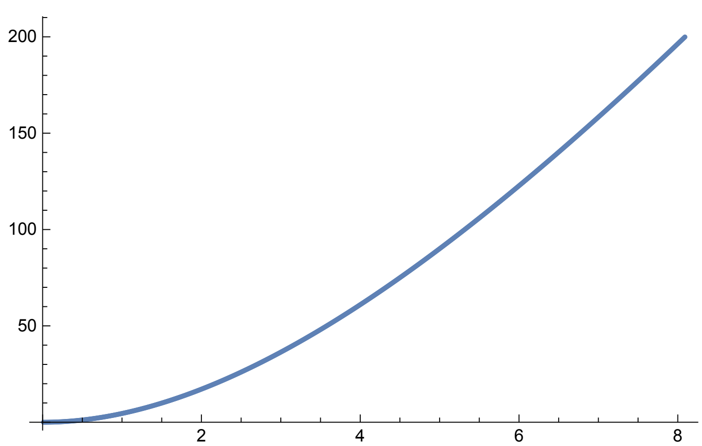
As $t \to \infty$, we see that $v \to m g/c$, and this is the terminal velocity. You could also determine terminal velocity by setting dv/dt = 0 in Newton' s second law.

b) Using Euler' s method we could write :

In[147]:=
```
Clear[y, v, h, c, m, g, nterms, initialheight]
y[0] = 0; v[0] = 0; h = 0.01; g = 9.8; m = 1; initialheight = 200; c = 0.2;
a[v_] := g - c v / m
v[n_] := v[n] = v[n - 1] + h a[v[n - 1]]
y[n_] := y[n] = y[n - 1] + h v[n - 1]

(* Now define a variable nterms to determine how many
 iterations will be required for the mass to reach the ground *)

nterms = Catch[Do[If[y[n] > initialheight, Throw[n - 1]], {n, 10 000}]];
ListPlot[Table[{n h, y[n]}, {n, nterms}]]
Print["Time of flight = ", nterms h,
 " s.  Velocity at impact = ", v[nterms], " m/s"]
```

Out[153]=



```
Time of flight = 8.09 s.  Velocity at impact = 39.2993 m/s
```

c) We know from basic kinematics that the time to fall a height H (in the absence of friction) is $\sqrt{2H/g}$

and the velocity at impact is $\sqrt{2 g H}$ In this case, those values are:

```
In[106]:= Print["Time of flight with no friction = ", Sqrt[2 initialheight / g],
  "  secs.  Velocity at impact with no friction = ",
  Sqrt[2 g initialheight], " m/s"]
```

```
Time of flight with no friction = 6.38877
   secs.  Velocity at impact with no friction = 62.6099 m/s
```

Now, if I set c = 0 in the initial program (I will post just the results and not list the entire program again), we get :

```
Time of flight = 6.39 Velocity at impact = 62.622
```

And we achieve good consistency.

d) I will construct a table of results below :

m (kg)	t (s)	v (m / s)
1	8.09	39.3
10	6.53	60
100	6.4	62.3

The clear trend is that as the mass increases, the final values approach those of free fall with no gravity. The reason for this is clear from Newton’s second law, as mass increases, the ratio of the frictional acceleration to the gravitational acceleration decreases; as this ratio decreases, the acceleration approaches g.

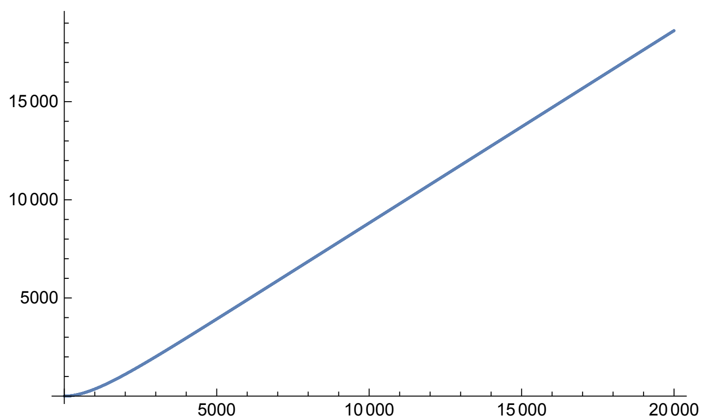
e) Now we want to know how far an object of m = 2 kg needs to fall to reach 99 % of its terminal velocity. I will rewrite the program making slight changes that will allow me to determine how many iterations are needed for the velocity to exceed 0.99*(m g/c) :

In[162]:=
```
Clear[y, v, h, c, m, g, nterms]
y[0] = 0; v[0] = 0; h = 0.01; g = 9.8; m = 2; c = 0.2;
a[v_] := g - c v / m
v[n_] := v[n] = v[n - 1] + h a[v[n - 1]]
y[n_] := y[n] = y[n - 1] + h v[n - 1]
ListPlot[Table[y[n], {n, 20 000}]]
(* Now define a variable nterms1 to determine how many iterations
 will be required for the velocity to exceed 0.99* (m g/c) *)

nterms1 = Catch[Do[If[v[n] > 0.99 * (m g / c), Throw[n - 1]], {n, 10 000}]];

Print[
 "Object achieves 99% of terminal velocity after falling for a distance of ",
 y[nterms1] " m and is traveling at ", v[nterms1], " m/s"]
```

Out[167]=




```
Object achieves 99% of terminal velocity after falling for a distance of
 3539.77 m and is traveling at 97.0191 m/s
```

This may seem like an unrealistically long distance to fall to approach terminal velocity; and it likely is. Objects this large with this much friction are much more likely to experience friction that varies as the square of the velocity. Try to redo this problem with quadratic friction and see what your results are; think about what other variables might have to be changed.