PHYS 301
HOMEWORK #12-- Solutions

1. We start with Hermite' s differential equation :

$$y'' - 2\,x\,y' + 2\,k\,y = 0$$

where k is an integer. We start with our trial solution:

$$y = \sum_{n=0}^{\infty} a_n\, x^n$$

which allows us to write:

$$y' = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} \quad \text{and } y'' = \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^{n-2}$$

Using these relations in the original differential equation yields:

$$\sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^{n-2} - 2\sum_{n=1}^{\infty} n\, a_n\, x^n + 2\,k \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Re - indexing the first sum :

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\, a_{n+2}\, x^n - 2\sum_{n=1}^{\infty} n\, a_n\, x^n + 2\,k \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Stripping out the n=1 terms in the first and last sums:

$$2\,a_2 + 2\,ka_o + \sum_{n=1}^{\infty} [(n+2)\,(n+1)\, a_{n+2} - 2\,n\,a_n + 2\,k\,a_n]\, x^n = 0$$

The stripped out terms tells us that $a_2 = -\,k\,a_o$, and the summation produces the recursion relation:

$$a_{n+2} = \frac{2\,(n-k)}{(n+2)\,(n+1)}\, a_o$$

Notice that the series truncates when n = k, and that there will be an odd and even branch of the solution.

b) We are given that a_o= 0 and a_1=15.

Since the recursion relation equates odd coefficients with each other, and even coefficients with each other, a_2 is a multiple of a_o (and so is zero); $a_4$is a multiple of a_2 (and so is zero), and by similar reasoning, we can show all even coefficients are zero.

For k = 5, we can see that when n = 5, the numerator will go to zero, meaning that a_7 is zero. Hence, our solution is an odd polynomial of order 5. We find the remaining coefficients:

$$a_3 = 2\,(1-5)\,\frac{a_1}{3\cdot 2} = \frac{-4}{6}\,a_1 = \frac{-2}{3}\cdot 15 = -\,20$$

$$a_5 = \frac{2\,(3-5)\,a_3}{5\cdot 4} = \frac{-\,a_3}{5} = 4$$

And the polynomial that results is:

$$y = 15\,x - 20\,x^3 + 4\,x^5$$

Test that this solutions satisfies the original ODE:

In[87]:=
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Clear[f]
f[x_] := 4 x^5 - 20 x^3 + 15 x
Simplify[f''[x] - 2 x f'[x] + 10 f[x]]
```

Out[89]= 0

If we solved the original differential equation with the given initital conditions using DSolve in *Mathematica*:

In[8]:=
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DSolve[{y''[x] - 2 x y'[x] + 10 y[x] == 0, y[0] == 0, y'[0] == 15}, y[x], x]
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Out[8]= $\{\{y[x] \to 15\,x - 20\,x^3 + 4\,x^5\}\}$

Verifying our solution.

2. We know that we can expand functions satisfying Dirichlet' s conditions on [-1, 1] in a Legendre series of the form :

$$f(x) = \sum_{m=0}^{\infty} c_m P_m(x)$$

where the P_m(x) are the Legendre polynomials and the coefficients are found using:

$$c_m = \frac{2\,m+1}{2} \int_{-1}^{1} f(x)\, P_m(x)\, dx$$

Setting m = 0, we find:

$$c_0 = \frac{1}{2} \int_{-1}^{1} \left(x^2 + 3\right) \cdot 1\, dx$$

Now, this is an easy integral to do, but we should keep up our good habits an notice that this is an even integrand on an interval [-L, L], so we can use symmetry and write this as :

$$c_0 = \int_{0}^{1} \left(x^2 + 3\right) dx = \frac{10}{3}$$

Setting m = 1 gives us:

$$c_1 = \frac{3}{2} \int_{-1}^{1} \left(x^2 + 3\right) P_1(x)\, dx$$

and this is again a simple integral, but if we think for just a second, we realize we don’t have to do the integral at all. We learned in class that the Legendre polynomials are even if m is even, and odd if m is odd. Thus, P_1 is odd, our function $\left(x^2 + 3\right)$ is even, so their product yields an odd integrand. And we know that the integral of an odd integrand over [-L,L] is zero, so we can conclude that c_1, c_3, and all other odd coefficients will be zero. In short, our Legendre series will consist of only even powers (which makes sense given that our function is even).

$$c_2 = \frac{5}{2}\int_{-1}^{1}\left(x^2+3\right)P_2(x)\,dx = 5\int_{0}^{1}\left(x^2+3\right)P_2(x)\,dx = \frac{5}{2}\int_{0}^{1}\left(x^2+3\right)\left(3x^2-1\right)dx = \frac{2}{3}$$

$$c_4 = \frac{9}{2}\int_{-1}^{1}\left(x^2+3\right)P_4(x)\,dx = 9\int_{-1}^{1}\left(x^2+3\right)\cdot\frac{1}{8}\left(35x^4-30x^2+3\right)dx = 0$$

We can write our Legendre series as:

$$f(x) = c_o P_o + c_2 P_2 = \frac{10}{3}+\frac{2}{3}\cdot\frac{1}{2}\left(3x^2-1\right) = \frac{10}{3}+x^2-\frac{1}{3} = x^2+3$$

3. We find the Legendre series for our old friend :

$$f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

We can use symmetry arguments to recognize that our function is odd, and therefore all the even coefficients will be zero. We can find the coefficients:

$$c_1 = \frac{3}{2}\int_{-1}^{1} f(x)P_1(x)\,dx = 3\int_{0}^{1} 1\cdot P_1(x)\,dx = \frac{3}{2}$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} f(x)P_3(x)\,dx = 7\int_{0}^{1} 1\cdot P_3(x)\,dx = 7\cdot\frac{1}{2}\int_{0}^{1}\left(5x^3-3x\right)dx = \frac{-7}{8}$$

We could continue this process on, let's see how *Mathematica* can help. Below is a short program which outputs the first ten coefficients and also plots the Legendre sum of the terms out to P_{31}:

In[82]:=
```
Clear[m, f, coefficient, x, y]
f[x_] := Which[−1 < x < 0, −1, 0 < x < 1, 1]
coefficient[m_] :=
  coefficient[m] = (2 m + 1) / 2 Integrate[f[x] LegendreP[m, x], {x, −1, 1}]
Plot[Sum[coefficient[n] LegendreP[n, y], {n, 0, 31}], {y, −1, 1}]
Do[Print["The ", m, " th Legendre coefficient is: ", coefficient[m]], {m, 0, 10}]
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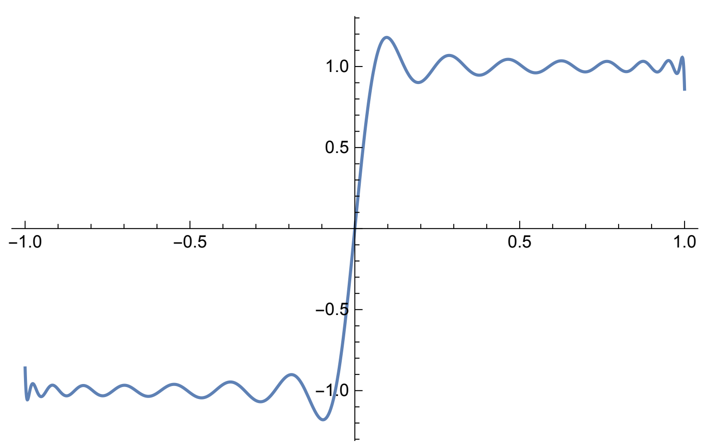
Out[85]=


The 0 th Legendre coefficient is: 0

The 1 th Legendre coefficient is: $\frac{3}{2}$

The 2 th Legendre coefficient is: 0

The 3 th Legendre coefficient is: $-\frac{7}{8}$

The 4 th Legendre coefficient is: 0

The 5 th Legendre coefficient is: $\frac{11}{16}$

The 6 th Legendre coefficient is: 0

The 7 th Legendre coefficient is: $-\frac{75}{128}$

The 8 th Legendre coefficient is: 0

The 9 th Legendre coefficient is: $\frac{133}{256}$

The 10 th Legendre coefficient is: 0

So the first several terms of the Legendre series are:

$$f(x) = \frac{3}{2} P_1 - \frac{7}{8} P_3 + \frac{11}{16} P_5 - \ldots$$

4. Following the discussion in class and in the classnote, we write the potential at O as :

$$V = \frac{2\,k\,q}{r} - \frac{k\,q}{r_1} - \frac{k\,q}{r_2} = k\,q\left(\frac{2}{r} - \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right)$$

Note carefully how the signs of the charges are included in this expression. We know from class that we can write $1/r_1$ and $1/r_2$ in terms of the law of cosines, and then use Maclaurin series to show that these terms generate the Legendre polynomials. Using results from class, we get:

$$\frac{1}{r_1} = \frac{1}{r} \sum_{m=0}^{\infty} P_m(\cos\theta)\,(a/r)^m$$

$$\frac{1}{r_2} = \frac{1}{r} \sum_{m=0}^{\infty} (-1)^m P_m(\cos\theta)\,(a/r)^m$$

Remember that the $(-1)^m$ arises from the $\cos(\pi\text{-}\theta)$ term in the distance expression for r_2.

The expression for potential becomes:

$$V = \frac{k\,q}{r}\left[2 - \left[\sum_{m=0}^{\infty} P_m(\cos\theta)\,(a/r)^m + \sum_{m=0}^{\infty} (-1)^m P_m(\cos\theta)\,(a/r)^m\right]\right]$$

Here, the odd terms in the series sum to zero, leaving us with even terms:

$$V = \frac{k\,q}{r}\left[2 - \left(2\,P_0\,(\cos\theta)\,(a/r)^0 + 2\,P_2\,(\cos\theta)\,(a/r)^2 + 2\,P_4\,(\cos\theta)\,(a/r)^4 + ...\right)\right]$$

We can simplify this expansion a bit more by focusing on the term $2\,P_0(\cos\theta)\,(a/r)^0$. Since P_0 is 1 and since $(a/r)^0$is also 1, the expansion simplifies to:

$$V = \frac{k\,q}{r}\left[2 - \left(2 + 2\,P_2\,(\cos\theta)\,(a/r)^2 + 2\,P_4\,(\cos\theta)\,(a/r)^4 + ...\right)\right]$$

$$= 2\,\frac{k\,q}{r}\sum_{m=2}^{\infty} P_m\,(\cos)\,(a/r)^m \quad \text{for even values of m.}$$

The leading term in the expansion is then

$$2\,\frac{k\,q}{r}\left(\frac{1}{2}\left(3\cos^2\theta - 1\right)(a/r)^2\right) = \frac{k\,q\,a^2}{r^3}\left(3\cos^2\theta - 1\right)$$

an equation familiar to you from electrodynamics, where q is the charge, a is the separation between charges, r (>>a) is the distance from the origin to the observer, θ is the angle between the observer and the quadrupole's center, and the constant k is $1/(4\,\pi\,\epsilon_o)$ where ϵ_o is the permittivity of free space.