PHYS 301
HOMEWORK #13

Solutions

1. Problem 11.82, text, all parts except b). 5 pts each part.

a) We start with

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + y = 0$$

with y (0, t) = y (1, t) = 0

assume y (x, t) = X(x)T(t) and substitute into the original equation:

$$X\,T'' - X''\,T + X\,T = 0$$

divide by the solution to separate variables :

$$\frac{T''}{T} - \frac{X''}{X} + 1 = 0$$

doing the requested choreography:

$$\frac{T''}{T} + 1 = \frac{X''}{X}$$

b) We have described several times in class why each side of the equation must equal a constant.

c) As shown in class, setting $P > 0$ leads to exponential solutions; setting $P = 0$ leads to a straight line, setting $P < 0$ leads to trig functions.Only trig functions can satisfy the boundary conditions stated above. Therefore the X equation yields :

$$\frac{X''}{X} = -k^2 \Rightarrow X'' + k^2 X = 0 \Rightarrow X = A\cos(k\,x) + B\sin(kx)$$

d & e) using the boundary conditions, we get :

$$y(0, t) = A\cos(0) + B\sin(0) = 0 \Rightarrow A = 0$$

$$y(1, t) = B\sin k = 0 \Rightarrow k = n\pi$$

f) The ODE for T is :

$$\frac{T''}{T} + 1 = -k^2 \Rightarrow \frac{T''}{T} = -1 - k^2 = -\left(1 + n^2\pi^2\right)$$

The solution to this ODE is:

$$T = C\cos\left[\left(\sqrt{1+n^2\pi^2}\right)t\right] + D\sin\left[\left(\sqrt{1+n^2\pi^2}\right)t\right]$$

g) Because our equation is homogeneous and our boundary conditions are homogeneous, we know that any sum of solutions (normal modes) will also be a solution. So we sum over all the normal modes to get :

$$y(x,t) = \sum_{n=1}^{\infty}\left[\sin(n\pi x)\left[C_n\left(\cos\sqrt{1+n^2\pi^2}\right)t\right] + D_n\sin\left[\left(\sqrt{1+n^2\pi^2}\right)t\right]\right]$$

if you are wondering where the coefficient B went, we simply combined it with C and D (remember, a constant times a constant is another constant, and we can call it anything we wish).

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2. Problem 11.84; start with:

$$\frac{\partial y}{\partial t} = c^2\left(\frac{\partial^2 y}{\partial x^2}\right) \text{ with } y(0,t) = y(L,t) = 0 \text{ and } y(x,0) = f(x)$$

Follow the drill; assume a trial solution of $y(x,t) = X(x)\,T(t)$

$$X\,T' = c^2 X''\,T$$

Then divide by the solution to separate variables :

$$\frac{1}{c^2}\frac{T'}{T} = \frac{X''}{X}$$

As we know by now, each side must be a constant. We move the constant c to the T side to make our trig functions simpler. Only sinusoidal functions match the boundary conditions, so we can write:

$$\frac{X''}{X} = -k^2 \Rightarrow X'' + k^2 X = 0 \Rightarrow X = A\cos(kx) + B\sin(kx)$$

The condition that $y(0,t) = 0$ tells us that $A = 0$; $y(L,t) = 0 \Rightarrow k L = 0 \Rightarrow k = n\pi/L$

Now we solve the T equation:

$$\frac{1}{c^2}\frac{T'}{T} = -k^2 \Rightarrow T' = -c^2 k^2 T$$

The solution to this is a simple exponential :

$$T = C\,e^{-c^2 k^2 t} = C\,e^{-c^2 n^2 \pi^2 t/L^2}$$

where $k = n\pi/L$

So we know our general solution, $y(x,t) = X(x)\,T(t)$ will be a product of trig functions times this exponential. We also know (review section 11.3) that there are an infinite number of normal modes that satisfy the original equation and the given boundary conditions. Since the equation is homogeneous, and the boundary conditions are homogeneous, the sum of solutions is also a solution, so that the total solution for this problem will involve this sum:

$$y(x, t) = \sum_{n=1}^{\infty}\left[B_n \sin(n\pi x/L)\, e^{-c^2 n^2 \pi^2 t/L^2}\right]$$

where we have combined constants into a single constant, B_n. We know apply the initial condition y(x,0)= f(x). Setting t = 0 in the general solution:

$$y(x, 0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) = f(x)$$

We recognize this immediately as a Fourier sine series. I can solve for the coefficients B_n by using the odd extension to make a function periodic on [-L,L]. (I have to use the odd extension because my basis set is the odd trig function, sin. If our basis set were cos, we would make the even extension.). Thus, the coefficients B_n are identically the Fourier sine coefficients:

$$b_n = \frac{2}{L}\int_0^L f(x) \sin(n\pi x/L)\, dx$$

We can express our complete solution as:

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)\, e^{-c^2 n^2 \pi^2 t/L^2}$$

$$\text{where } B_n = \frac{2}{L}\int_0^L f(x) \sin(n\pi x/L)\, dx$$

We cannot go beyond this point unless we are given a specific functional form for f(x).

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3. Problem 11.91. Let's start with the heat diffusion equation in one dimension:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

with u(0,t) = u(L,t) = 0 and u(x,0) = u_o sin(2 π x/L), where u is temperature, t is time, α is related to the thermal properties of the material (we use u to denote temperature to avoid confusion with time).

We work the program: Insert the trial solution u(x,t) = X(x) T(t) into the original PDE and obtain:

$$X T' = \alpha X'' T$$

divide by the solution to separate variables:

$$\frac{1}{\alpha}\frac{T'}{T} = \frac{X''}{X}$$

(We could leave α on the X side, but moving it to the T side makes our expressions a little easier to deal with).

Since the LHS depends only T and the RHS depends only on X, we know that each side equals a constant. Because our boundary conditions are u(0,t) = u(0,L)=0, we know from vast prior experience that the constant needs to be a negative number to generate trig functions to match the boundary conditions. This means that our X equation has the solution:

$$X = A \cos kx + B \sin kx$$

The condition that u (0, t) = 0 ⇒ A = 0 and u (L, t) = 0 ⇒ k = n π/L so that our X solution is :

$$X = B \sin (n \pi x / L)$$

Solving the T equation:

$$\frac{1}{\alpha} \frac{T'}{T} = -k^2 \Rightarrow T' = -\alpha k^2 T \Rightarrow T = C e^{-\alpha k^2 t} = C e^{-\alpha n^2 \pi^2 t/L^2}$$

The complete solution will be the sum over all normal modes:

$$u (x, t) = \sum_{n=1}^{\infty} B_n \sin (n \pi x / L) e^{-\alpha n^2 \pi^2 t/L^2}$$

We know that now we set t = 0 and use the initial condition:

$$u (x, 0) = \sum_{n=1}^{\infty} B_n \sin (n \pi x / L) = u_o \sin (2 \pi x / L)$$

Our instinct would be to take the Fourier integral to compute the coefficients, but let’s take a moment to look at our initial condition again. Notice that the initial condition ($u(x,0) = u_o \sin(2 \pi x/L)$) is a normal mode of the equation. In this case, the only non-zero coefficient will be n =2, when $B_2 \sin(2 \pi x/L) = u_o \sin(2 \pi x/L)$. The coefficients vanish for all other values of n. (You would obtain the same result if you did the Fourier integral:

$$B_n = \frac{2}{L} \int_0^L u_o \sin (2 \pi x / L) \sin (n \pi x / L) dx$$

Thus, there is only one term in the solution (the n = 2 term), and we can write :

$$u (x, t) = u_o \sin (2 \pi x / L) e^{-4 \alpha \pi^2 t/L^2}$$

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4. Consider a semi-infinte plate of width 10 cm; the temperature on the vertical boundaries is zero and the temperature across the bottom edge is T(x,0) = x. Find the solution to Laplace’s equation for this plate.

We are solving:

$$\nabla^2 T = 0$$

in two dimensions, or

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Our trial solution of T(x,y) = X(x)Y(y) yields:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

The boundary conditions that T(0,y) = T(10,y) =0 require that:

$$\frac{X''}{X} = -k^2 \Rightarrow X = A\cos kx + B\sin kx$$

From past experience we know that the T(0,y) = 0 condition requires that A = 0, and T(10,y) = 0 requires that k = n π/10.

The Y equation then satisfies:

$$\frac{Y''}{Y} = +k^2 \Rightarrow Y = C\,e^{k\,y} + D\,e^{-ky}$$

The imbedded boundary condition is that the temperature must remain finite as y grows large; this condition requires that C = 0, leaving us with the solution:

$$T(x, y) = X(x)\,Y(y) = B\sin(k\,x)\,e^{-k\,y} \text{ where } k = n\,\pi/10$$

Our complete solution is the sum of all the normal modes:

$$T(x, y) = \sum_{n=1}^{\infty} B_n \sin(n\,\pi\,x/10)\,e^{-n\,\pi\,y/10}$$

Employing the inhomogeneous boundary condition:

$$f(x, 0) = x = \sum_{n=1}^{\infty} B_n \sin(n\,\pi\,x/10)$$

We recognize this as a Fourier sine series and the coefficients B_n are identically the Fourier coefficients:

$$b_n = B_n = \frac{2}{10}\int_0^{10} x\sin(n\,\pi\,x/L)\,dx$$

which we find from:

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Simplify[Integrate[x Sin[n π x / 10], {x, 0, 10}], Assumptions → n ∈ Integers] (2 / 10)
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$$-\frac{20\,(-1)^n}{n\,\pi}$$

which allows us to write the solution as:

$$T(x, y) = \frac{20}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(n\,\pi\,x/10)\,e^{-n\,\pi\,y/10}}{n}$$

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5. Consider a semi - infinite plate of width π. The vertical sides are at T = 0 and the bottom edge satisfies T (x, 0) = cos x. Find the solution to Laplace' s equation for this plate.

Let' s make use of previous results and write down our solution as :

$$T(x, y) = \sum_{n=1}^{\infty} B_n \sin(n\,x)\,e^{-n\,y}$$

If you are at all unsure about this solution, please review the earlier solutions carefully. Now we apply the boundary condition:

$$T(x, 0) = \cos x = \sum_{n=1}^{\infty} B_n \sin(n x)$$

As before, we find the coefficients by making the odd extension to our function (f(x) = cos x, and we use the odd extension since we are expanding in a sine series):

$$B_n = b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(n x)\, dx$$

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Simplify[2 Integrate[Cos[x] Sin[n x], {x, 0, π}], Assumptions → n ∈ Integers]
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$$\frac{2\left(1+(-1)^n\right) n}{-1+n^2}$$

We can see that the coefficients are zero when n is odd, and when n is even we get:

$$B_n = \frac{4 n}{n^2 - 1}, \ \text{n even}$$

so our solution is:

$$T(x, y) = \sum_{n=\text{even}}^{\infty} \frac{4 n \sin(n x)\, e^{-n y}}{n^2 - 1}$$