PHYS 301
HOMEWORK #14

Due : 2 May 2016

This is an optional homework, your grade will not be effected if you do not submit this homework. If you do submit it, turn it in no later than 5 pm in my office (Cudahy 404). Your grade on this homework will replace your lowest homework score of the semester.

1. Problem 11.94

Let' s think about this problem physically. If a linear bar starts out at the same temperature everywhere, then there is no temperature gradient and there is no reason for the temperature to change at all. Therefore, we should expect that the temperature will always be described as u (x, t) = 0.

Now, let' s solve it in a more rigorous way. We start with the equation of heat diffusion :

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Substituting the trial solution u(x,t) = X(x)T(t)

$$X\,T' = \alpha\, X''\,T$$

Dividing by the solution separates the variables:

$$\frac{T'}{T} = \alpha \frac{X''}{X}$$

It will be algebraically easier to move the α to the T side, so we obtain:

$$\frac{X''}{X} = -k^2 \text{ (only the trig solutions will satisfy the boundary conditions)}$$

so that X = A cos kx + B sin kx. The condition that u(0,t) = 0 implies that A = 0, and u(L,t) = 0 requires that k = n π/L. The T solution becomes:

$$\frac{1}{\alpha}\frac{T'}{T} = -k^2 \Rightarrow T' = -\alpha\, n^2 \pi^2 t \big/ L^2$$

which has the solution :

$$T = C\, e^{-\alpha\, n^2 \pi^2 t/L^2}$$

Summing over the normal modes of u(x,t) = X(x)T(t):

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x / L)\, e^{-\alpha\, n^2 \pi^2 t/L^2}$$

Now, we set t = 0 and employ the initial condition, u (x, 0) = 0 :

$$u(x, 0) = 0 = \sum_{n=1}^{\infty} C_n \sin(n\pi x / L)$$

Things might seem anti-climactic at this point; all the C coefficients will be zero, which means that u(x,t) = 0, the trivial solution.

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2. Problem 11.96

This problem is very similar to problem #4 of HW 13. The principal difference is that the plate has a finite length (it is not semi-infinite). Also, the bottom edge is at zero and the top edge is held at a temperature u_0. If necessary, refer to problem 3, HW 13 to review why the general solution is:

$$T(x, y) = (A\cos kx + B\sin kx)\left(C e^{ky} + D e^{-ky}\right)$$

Unlike the earlier problem, we cannot just set C = 0 since the plate is not infinite in length. We are familiar with the boundary conditions on the vertical sides, and they require that A = 0 and k = n π / x_f (where x_f is the width of the plate). This allows us to write the solution in the form:

$$T(x, y) = \sin(n\pi x / x_f)\left(C e^{n\pi y/x_f} + D e^{-n\pi y/x_f}\right)$$

The boundary condition on the bottom edge, T(x,0) = 0 implies:

$$T(x, 0) = \sin(n\pi x / x_f)(C + D) = 0 \Rightarrow C = -D$$

Substituting this into the solution gives:

$$T(x, y) = C\sin(n\pi x / x_f)\left(e^{n\pi y/x_f} - e^{-n\pi y/x_f}\right) = C\sin(n\pi x / x_f)\sinh(n\pi y / x_f)$$

Summing over all normal modes:

$$T(x, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi x / x_f)\sinh(n\pi y / x_f) \tag{1}$$

Applying the boundary condition on the upper edge:

$$T(x, y_f) = u_o = \sum_{n=1}^{\infty} C_n \sin(n\pi x / x_f)\sinh(n\pi y_f / x_f)$$

Note carefully the subscripts in the sinh expression. We recognize this as the Fourier sine series for u_o, and the Fourier coefficient will be equal to C_n sinh(n π y_f / x_f). Solving for the Fourier coefficient we get:

$$b_n = \sinh(n\pi y_f)/ x_f C_n = \frac{2}{x_f}\int_0^{x_f} u_o \sin(n\pi x / x_f)\,dx = \frac{2 u_o}{x_f}\left[-\frac{x_f}{n\pi}\cos(n\pi x / x_f)\right|_0^{x_f}$$

$$= \frac{-2 u_o}{n\pi}[\cos(n\pi) - 1] = \begin{cases} 0, & n \text{ even} \\ 4 u_0 / n\pi, & n \text{ odd} \end{cases}$$

This means that the C_n coefficients that we need in eq. (1) are:

$$C_n = \frac{b_n}{\sinh(n\pi y_f / x_f)} = \frac{4 u_o}{n\pi \sinh(n\pi y_f / x_f)}$$

and our complete solution is:

$$T(x, y) = \frac{4 u_o}{\pi} \sum^{\infty} \frac{\sin(n \pi x / x_f) \sinh(n \pi y / x_f)}{n \sinh(n \pi y_f / x_f)}, \ n \text{ odd}$$

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3. Problem 11.146 (submit any Mathematica output with your answers)

a) The potential inside a sphere of radius a is given by (see text 11.7.3):

$$V(r, \theta) = \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta)$$

where the coefficients A_m are defined by

$$A_m = \frac{2m+1}{2 a^m} \int_0^{\pi} V(\theta) P_m(\cos\theta) \sin\theta \, d\theta$$

(I have also denoted these coefficients as c_m in class. The integral above is the same as $\int_{-1}^{1} V(x) P_m(x) \, dx$ when you set $x = \cos\theta$ (so $dx = -\sin\theta \, d\theta$ and be careful also to transform also the limits of integration.)

We are given the potential function on the surface of the sphere of radius a :

$$V(\theta) = \begin{cases} c, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi \end{cases}$$

(We can also write this as a function of x. If we make use of the substitution $x = \cos\theta$, we can write this as a function of x) :

$$V(x) = \begin{cases} c, & 0 < x < 1 \\ 0, & -1 < x < 0 \end{cases}$$

and our coefficients become simply :

$$A_m = \frac{2m+1}{2 a^m} \int_0^1 c \, P_m(x) \, dx$$

We can compute the coefficients (and I will print out only the first eleven):

```
Clear[A, c, a]
A[m_] := A[m] = (2 m + 1) / (2 a^m) Integrate[c LegendreP[m, x], {x, 0, 1}]
Do[Print[A[m]], {m, 0, 10}]
```

$\frac{c}{2}$

$\frac{3\,c}{4\,a}$

0

$-\frac{7\,c}{16\,a^3}$

0

$\frac{11\,c}{32\,a^5}$

0

$-\frac{75\,c}{256\,a^7}$

0

$\frac{133\,c}{512\,a^9}$

0

(We would have gotten the same results for these coefficients if we integrated over angle; printint out only the first five coefficients (note carefully the limits of integration)) :

```
Clear[a, A, c]
A[m_] := (2 m + 1) / (2 a^m) Integrate[c LegendreP[m , Cos[θ]] Sin[θ], {θ, 0, π / 2}]
Do[Print[A[m]], {m, 0, 4}]
```

$\frac{c}{2}$

$\frac{3\,c}{4\,a}$

0

$-\frac{7\,c}{16\,a^3}$

0

Getting back to the problem The first few terms of the Legendre expansion are then:

$$V(r,\ \theta) = \frac{c}{2} P_o(\cos\theta) + \frac{3\,c}{4}\left(\frac{r}{a}\right) P_1(\cos\theta) - \frac{7\,c}{16}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11\,c}{32}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) + ...$$

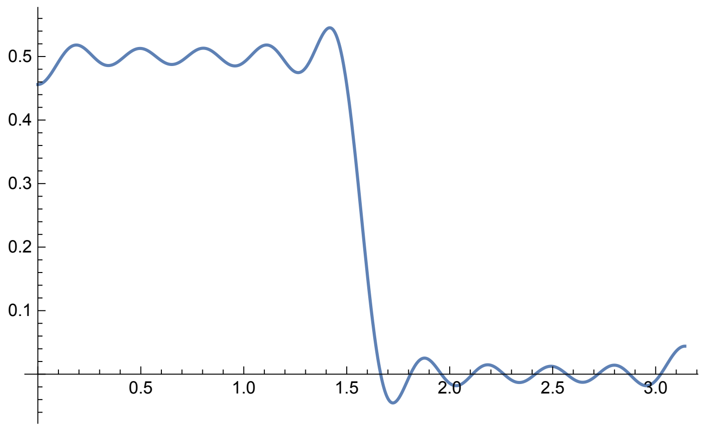
Remember that this is the expansion for the potential inside the sphere, so $r/a \leq 1$.

b) At the surface, r = a, so the potential can be written:

$$V(a, \theta) = \sum_{m=0}^{\infty} A_m\, a^m\, P_m(\cos\theta)$$

To plot the potential, we will need to provide a specific numerical value for c. We can write this as:

```
Clear[v, a, c]
c = 0.5;
v[θ_] := v[θ] = Sum[A[m] a^m LegendreP[m, Cos[θ]], {m, 0, 20}]
Plot[v[θ], {θ, 0, π}]
```

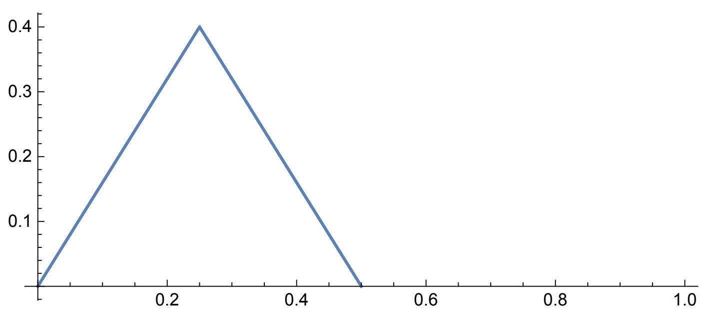


And you can see that this function matches the surface potential well; the potential is 0.5 for $0<\theta<\pi/2$ and zero for the second half of the interval. Since we set r = a, all the $(r/a)^m$ terms are equal to 1.

---

4. A string of length L is fixed at the ends and has zero initial velocity. Its initial position is given by :

$$y(x, 0) = \begin{cases} 4hx/L, & 0 < x < L/4 \\ 2h - 4hx/L, & L/4 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$



(The graph above used the specific values h = 0.4 and L = 1; this was necessary to get Mathematica to produce a graph, your answers should just use the variables h and L). Solve the wave equation for this set of boundary and initial conditions.

We worked in detail in class the general solution to the wave equation in Cartesian coordinates (see eq 11.4.14 in the text). Having done all that work, we can make use of that result here. All we need to do are find the values of the coefficients C_n and D_n using our initial conditions for the shape and velocity of the string. In eq. 11.4.14, g(x) is the initial velocity of the string, and f(x) is the initial shape. Since we are told the string starts at rest, g(x) =0 thus all the C_n are zero. We find D_n by setting t = 0 in the general solution and applying the boundary condition:

$$y(x, 0) = \sum_{n=1}^{\infty} D_n \sin(n\pi x/L) = f(x)$$

Thus, we can find the D_n coefficients by computing the Fourier integral :

$$D_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\,dx$$

For the function given, we obtain:

$$D_n = \frac{2}{L}\left[\int_0^{L/4}\frac{4hx}{L}\sin(n\pi x/L)\,dx + \int_{L/4}^{L/2}(2h - 4hx/L)\sin(n\pi x/L)\,dx\right]$$

We can use *Mathematica* to find that:

$$D_n = \frac{64h\cos(n\pi/8)\sin^3(n\pi\ /8)}{n^2\pi^2}$$

And the complete solution is:

$$y(x, t) = \frac{64h}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi/8)\sin^3(n\pi/8)\sin(n\pi x/L)\cos(n\pi v t/L)}{n^2}$$

To see how the string's shape evolves over time, type the following code into your notebook and view using a slow speed:

```
Clear[h, L, d, v]
(*We have to provide some arbitrary but reasonable
 values to allow Mathematica to produce a plot *)
h = 0.4; L = 1; v = 2;
d[n_] := 64 h Cos[n π / 8] Sin[n π / 8]^3 / (n^2 π^2)
Manipulate[
 Plot[Sum[d[n] Sin[n π x / L] Cos[n π v t / L], {n, 1, 21}], {x, 0, L}], {t, 0, 4 π}]
```

The next two problems are very similar to each other and also to #3 on this assignment. We use exactly the same equations and techniques to find the temperature inside the sphere as we did to find the potential (since both T and V satisfy Laplace's equation.

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5. Find the steady state temperature inside a sphere of radius 1 if the surface temperature is 35 $\cos^4\theta$ (assume azimuthal symmetry).

We can express the boundary condition here as :

$$T(1, \theta) = 35\cos^4\theta = 35x^4 \text{ where } x = \cos\theta$$

Using the solution to Laplace's equation in spherical coordinates, we can write the internal temperature is:

$$T(r,\ \theta)\ =\ \sum_{m=0}^{\infty} A_m\, r^m\, P_m(\cos\theta)$$

where the coefficients A_m are given by:

$$A_m = \frac{2\,m+1}{2\,a^m}\int_0^1 35\,x^4\,P_m(x)\,dx$$

Since our boundary condition is an even function, we expect that our expansion will consist only of even terms; all the odd coefficients will be zero. Setting the radius a = 1, we find the first several coefficients:

```
Clear[A]
A[m_] := (2 m + 1) / 2 Integrate[35 x^4 LegendreP[m, x], {x, -1, 1}]
Do[Print[A[m]], {m, 0, 6}]
```

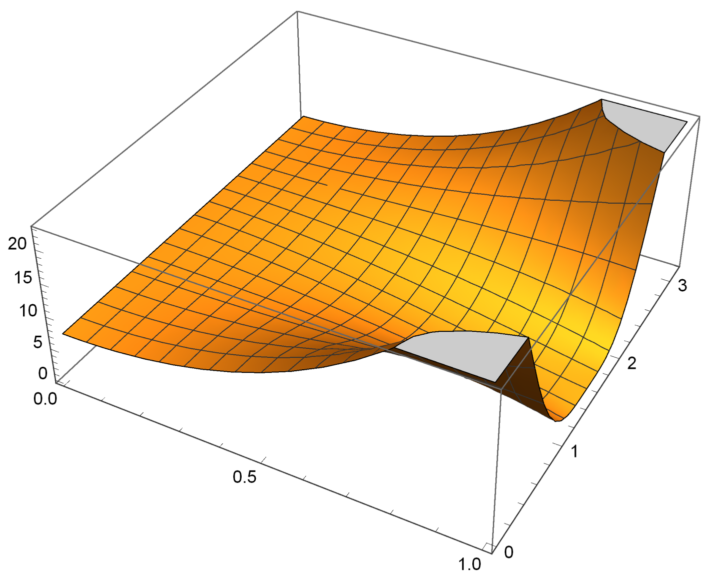
```
7

0

20

0

8

0

0
```

We can see that all the odd coefficients are zero, and the Legendre expansion truncates at the m = 4 term; we can write the interior temperature as the series (remember that a = 1):

$$T(r,\ \theta)\ =\ A_0\,r^0\,P_o(\cos\theta) + A_2\,r^2\,P_2(\cos\theta) + A_4\,r^4\,P_4(\cos\theta)$$
$$=\ 7 + 20\,r^2\,\frac{\left(3\cos^2\theta\ -1\right)}{2} + 8\,r^4\,\frac{\left(35\cos^4\theta\ -\ 30\cos^2\theta + 3\right)}{8}$$

We can try to see what the temperature distribution looks like by using Plot3D:

```
Clear[temp, r, θ]
temp[r_, θ_] := 7 + 10 r^2 (3 Cos[θ]^2 - 1) + r^4 (35 Cos[θ]^4 - 30 Cos[θ]^2 + 3)
Plot3D[temp[r, θ], {r, 0, 1}, {θ, 0, π}]
```



The radius is plotted along the lower edge, θ along the right edge, and temperature is the vertical scale on the left. Keep in mind we are mapping a sphere to a rectangular plot.

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6. Find the steady state temperature inside a sphere of radius 1 if the surface temperature is :

$$T(1, \theta) = \begin{cases} 100, & 0 < \theta < \pi/3 \\ 0, & \text{otherwise} \end{cases}$$

This boundary condition means that the temperature is 100 everywhere on the surface between the North Pole and latitude 60^oN, and zero everywhere south of that. We find our coefficients from the integral:

$$A_m = \frac{(2m+1)}{2} \int_{1/2}^{1} 100 \cdot P_m(x)\, dx$$

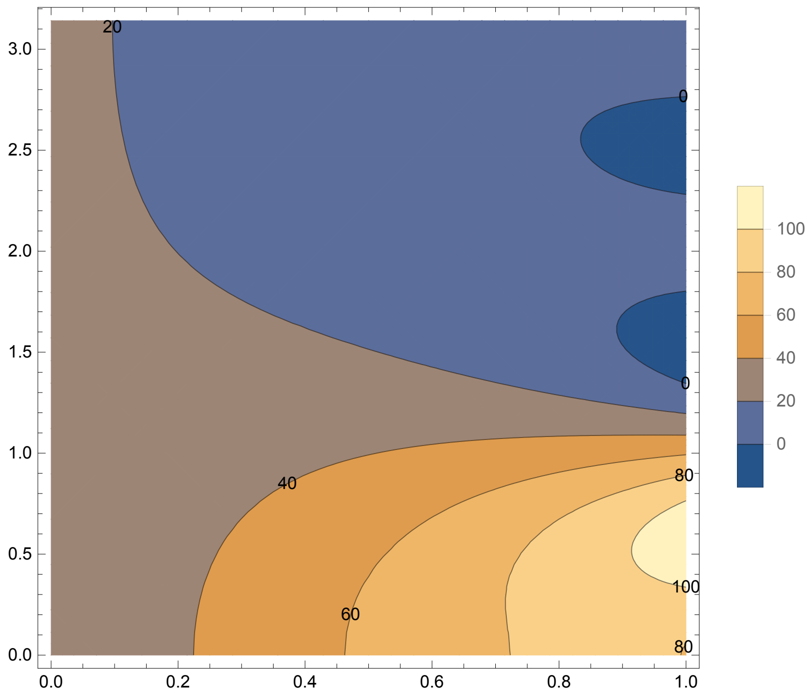
These are our limits of integration since $\cos 0 = 1$ and $\cos \pi/3 = 1/2$ We can print out the first six terms of the temperature

```
Clear[A, r]
A[m_] := (2 m + 1) / 2 Integrate[100 LegendreP[m, x], {x, 1/2, 1}]
series = Sum[A[m] r^m LegendreP[m, Cos[θ]], {m, 0, 5}];
Print[series]
```

$$25 + \frac{225}{4} r \cos[\theta] + \frac{375}{16} r^2 \left(-1 + 3 \cos[\theta]^2\right) + \frac{525}{128} r^3 \left(-3 \cos[\theta] + 5 \cos[\theta]^3\right) - \frac{3375\, r^4 \left(3 - 30 \cos[\theta]^2 + 35 \cos[\theta]^4\right)}{1024} - \frac{15\,675\, r^5 \left(15 \cos[\theta] - 70 \cos[\theta]^3 + 63 \cos[\theta]^5\right)}{4096}$$

Notice that the temperature is 25^o at the center of the sphere; we can try to visualize the temperature using a two dimensional contour plot:

```
ContourPlot[series, {r, 0, 1}, {θ, 0, π},
 PlotLegends → Automatic, ContourLabels → True]
```



In the contour plot above, the radius is along the bottom edge, and the polar angle, measured in radians, is on the left vertical axis.