

PHYS 301

HOMEWORK #3

Solutions

1. For solutions to this problem, read the discussion of celestial mechanics (which also shows the details of deriving conic sections from the solution to the differential equation).
2. Let's start by considering the cos addition formula :

$$\cos(m+n)x = \cos mx \cos nx - \sin mx \sin nx$$

$$\cos(m-n)x = \cos mx \cos nx + \sin mx \sin nx$$

If we add these equations, we get :

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$$

If we subtract these equations, we get :

$$\sin(mx) \sin(nx) = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

We can see immediately that our integrals can be written as :

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m-n)x + \cos(m+n)x] dx \\ \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] dx\end{aligned}$$

These integrate very easily to :

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx &= \frac{1}{2} \left[\frac{1}{m-n} \sin(m-n)x \Big|_{-\pi}^{\pi} + \frac{1}{m+n} \sin(m+n)x \Big|_{-\pi}^{\pi} \right] \\ \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx &= \frac{1}{2} \left[\frac{1}{m-n} \sin(m-n)x \Big|_{-\pi}^{\pi} - \frac{1}{m+n} \sin(m+n)x \Big|_{-\pi}^{\pi} \right]\end{aligned}$$

Now, for the case where $m \neq n$, we are left with a series of terms where we evaluate $\sin(p\pi)$ where p is always an integer. We know that $\sin(p\pi) = 0$ for all integer values of p , so we can see that these integrals equal zero in the case where $m \neq n$.

For the case where $m = n$, we can either apply L'Hopital's rule to these expressions, or go back to our original integrals and consider :

$$\int_{-\pi}^{\pi} \cos(m x) \cos(n x) dx = \int_{-\pi}^{\pi} \cos^2(m x) dx$$

or similarly,

$$\int_{-\pi}^{\pi} \sin(m x) \sin(n x) dx = \int_{-\pi}^{\pi} \sin^2(m x) dx$$

This is an integral you likely did in first year calc; use the substitutions

$\cos(2x) = \cos^2 x - \sin^2 x$ along with $\cos^2 x + \sin^2 x = 1$ to obtain :

$$\cos(2x) = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1}{2}(\cos 2x + 1)$$

Thus,

$$\int_{-\pi}^{\pi} \cos^2 x dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos(2x) + 1) dx = \frac{\sin 2x}{4} \Big|_{-\pi}^{\pi} + \frac{x}{2} \Big|_{-\pi}^{\pi} = \pi$$

Following the same procedure will produce the same result for :

$$\int_{-\pi}^{\pi} \sin^2 x dx = \pi$$

To compute the $\int \sin(m x) \sin(n x) dx$ integral, we use the sin addition law:

$$\sin(m+n)x = \sin(m x) \cos(n x) + \sin(n x) \cos(m x)$$

$$\sin(m-n)x = \sin(m x) \cos(n x) - \sin(n x) \cos(m x)$$

Subtract equations to get :

$$\sin(m x) \cos(n x) = \frac{1}{2}[\sin(m+n)x - \sin(m-n)x]$$

Our integral becomes :

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(m x) \cos(n x) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\sin(m+n)x - \sin(m-n)x] dx \\ &= \frac{1}{2} \left[\frac{-1}{m+n} \cos(m+n)x \Big|_{-\pi}^{\pi} + \frac{1}{m-n} \cos(m-n)x \Big|_{-\pi}^{\pi} \right] \end{aligned}$$

Now we are evaluating cos between π and $-\pi$; since cos is an even function, we know that $\cos(a\pi) = \cos(-a\pi)$ where a is any coefficient, hence we can conclude at this point that this integral is always zero, even if $m = n$. To see this last result, consider

$$\int_{-\pi}^{\pi} [\sin(m+n)x - \sin(m-n)x] dx$$

when $m = n$. The first part of the integral produces a result of zero (see paragraph above). The second part produces zero since $\sin(m-n) = \sin(0) = 0$ when $m = n$.

3. a) We start with Euler's formula:

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

Adding equations gives :

$$e^{ix} + e^{-ix} = 2 \cos x \Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Subtract the top two equations and obtain:

$$e^{ix} - e^{-ix} = 2i \sin x \Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\text{b) } \int_{-\pi}^{\pi} e^{ikx} dx = \frac{1}{ik} (e^{ik\pi} - e^{-ik\pi}) = 2 \sin k\pi = 0$$

since we know that $\sin(n\pi) = 0$ for all integer values of n .

4. Now we write the integrals using exponentials :

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx &= \int_{-\pi}^{\pi} \frac{(e^{imx} - e^{-imx})}{2i} \frac{(e^{inx} - e^{-inx})}{2i} dx \\ &= \int_{-\pi}^{\pi} \frac{e^{i(m+n)x} - e^{i(m-n)x} - e^{i(n-m)x} + e^{-i(m+n)x}}{4i^2} dx \end{aligned}$$

When $m \neq n$, each of the terms goes to zero since we have already shown that the integral of

$$\int_{-\pi}^{\pi} e^{ikx} dx$$

is zero for any integer value of k . In the case where $n = m$, the first and last terms still yield zero integrals, however, the middle two terms are -1 giving us :

$$\int_{-\pi}^{\pi} \frac{-2}{4i^2} dx = \pi \quad (\text{since } i^2 = -1)$$

For the cos integral, we proceed similarly:

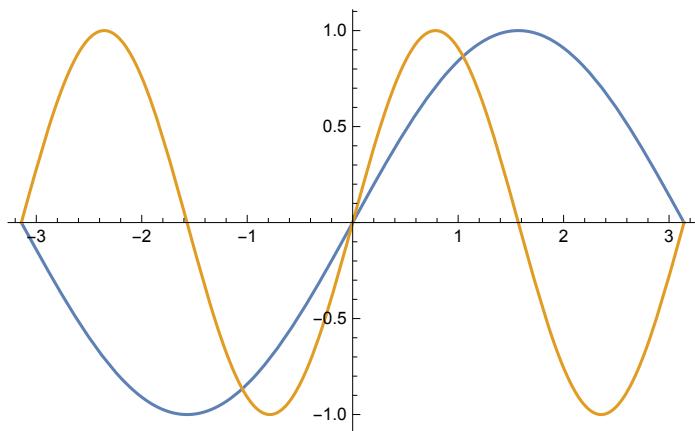
$$\begin{aligned} \int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx &= \int_{-\pi}^{\pi} \frac{(e^{imx} + e^{-imx})}{2} \frac{(e^{inx} + e^{-inx})}{2} dx \\ &= \int_{-\pi}^{\pi} \frac{e^{i(m+n)x} + e^{i(m-n)x} + e^{i(n-m)x} + e^{-i(m+n)x}}{4} dx \end{aligned}$$

when n and m are unequal, each term yields a zero integral. When $n = m$, the first and fourth terms produce zero integrals, and the middle terms yield :

$$\int_{-\pi}^{\pi} \frac{2}{4} dx = \pi$$

5. We use Mathematica to plot :

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Plot[{Sin[x], Sin[2 x]}, {x, -π, π}]
```



If we plot the product of the curves and make use of some *Mathematica*, we can see clearly why the integral of $\sin x \sin 2x$ over this interval is zero:

```
g1 = Plot[Sin[x] Sin[2 x], {x, -π/2, π/2}, Filling → Axis, FillingStyle → Red];
g2 = Plot[Sin[x] Sin[2 x], {x, -π, -π/2}, Filling → Axis, FillingStyle → Green];
g3 = Plot[Sin[x] Sin[2 x], {x, π/2, π}, Filling → Axis, FillingStyle → Green];
Show[g1, g2, g3, PlotRange → All]
```

