## PHYS 301 HOMEWORK #4

## Solutions

1. We begin here with the expression for acceleration in spherical polar coordinates :

$$\mathbf{a} = \left(\ddot{\mathbf{r}} - \mathbf{r}\,\dot{\theta}^2 - \mathbf{r}\,\dot{\phi}^2\sin^2\theta\right)\mathbf{\hat{r}} \\ + \left(\mathbf{r}\,\ddot{\theta} + 2\,\dot{\mathbf{r}}\,\dot{\theta} - \mathbf{r}\,\dot{\phi}^2\sin\theta\cos\theta\right)\mathbf{\hat{\theta}} \\ + \left(\mathbf{r}\,\ddot{\phi}\sin\theta + 2\,\dot{\mathbf{r}}\,\dot{\phi}\sin\theta + 2\,\mathbf{r}\,\dot{\theta}\,\dot{\phi}\cos\theta\right)\mathbf{\hat{\phi}}$$

Also, remember that  $\theta$  is the polar angle (the angle measured down from the north pole, i.e., 90 - latitude) and that  $\phi$  is the azimuthal angle, which is essentially the longitude of the particle.

Since the particles are constrained to move on the edges of rings of fixed radii, we can set all time derivatives of r equal to zero, since radius is always a constant. We are also told that all angular velocities are constant, therefore and second derivatives of  $\theta$  and  $\phi$  will be zero.

The blue and green rings are rotating both in the  $\theta$  and  $\phi$  directions; let's write the expression for the acceleration of a particle on the blue ring and use the subscript B to designate the properties of the blue ring; the expression for the green ring will be identical except you would use a different subscript.

The acceleration of a particle on the blue ring is then:

$$\mathbf{a} = \left(-\mathbf{r}_{\mathrm{B}} \dot{\theta}_{\mathrm{B}}^{2} - \mathbf{r}_{\mathrm{B}} \dot{\phi}_{\mathrm{B}}^{2} \sin^{2} \theta\right) \mathbf{\hat{r}} + \left(-\mathbf{r}_{\mathrm{B}} \dot{\phi}_{\mathrm{B}}^{2} \sin \theta \cos \theta\right) \mathbf{\hat{\theta}} + \left(2\mathbf{r}_{\mathrm{B}} \dot{\theta}_{\mathrm{B}} \dot{\phi}_{\mathrm{B}} \cos \theta\right) \mathbf{\hat{\phi}}$$

The expression for acceleration on the pink ring is even simpler since the pink ring is rotating only in the azimuthal direction, meaning that all time derivates of theta will be zero. Thus, the expression for acceleration for a particle on the pink ring is :

$$\mathbf{a} = -\mathbf{r}_{\mathrm{P}} \,\dot{\phi}_{\mathrm{P}}^2 \sin^2\theta \,\,\hat{\mathbf{r}} \,-\mathbf{r}_{\mathrm{P}} \,\dot{\phi}_{\mathrm{P}}^2 \sin\theta\cos\theta \,\hat{\boldsymbol{\theta}}$$

At the North Pole,  $\theta=0$  and sin  $\theta=0$ , so the acceleration at the North Pole is zero.

2. a) 
$$f(x) = x$$

The function is odd on  $[-\pi, \pi]$ , so we know that all the a coefficients will be zero by symmetry. The b coefficients are found from :

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(n x) \, dx$$

Since both x and sin(n x) are odd, their product is even and we have the integral of an even integrand between  $-\pi$  and  $\pi$ . We can use this symmetry to rewrite the integral as shown below, and employ integration by parts to obtain:

$$b_{n} = \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) dx = \frac{2}{\pi} \left[ \frac{-1}{n} x \cos(nx) \Big|_{0}^{\pi} - \left( \frac{-1}{n} \int_{0}^{\pi} \cos(nx) dx \right) \right]$$

At this point we should be able to look at the last integral on the right and recognize that the result will involve sin(n x) evaluated at  $x = \pi$  and x = 0. Since  $sin 0 = sin(n \pi) = 0$  for integer values of n, we can conclude that the last integral on the right is zero, leaving us with:

$$b_n = \frac{-2}{\pi n} \left( \pi \cos \left( n \, \pi \right) \, - \, 0 \right) = \, \frac{-2}{n} \left( -1 \right)^n \, = \, \frac{2}{n} \left( -1 \right)^{n+1}$$

(Note carefully that the factor of x causes the term to go to zero when evaluated at x = 0.) And the Fourier series is:

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

We can verify this result via Mathematica:

$$g1 = Plot[(2) Sum[(-1)^{(n+1)} Sin[n x]/n, \{n, 1, 31\}], \{x, -\pi, \pi\}];$$
  

$$g2 = Plot[x, \{x, -\pi, \pi\}];$$
  
Show[g1, g2]



b) f (x) = Abs[x]The graph of this function is:



We can see that the function is clearly even on  $[-\pi,\pi]$  so we can use symmetry to set the b coefficients to zero, and we can write the a coefficients as:

$$\begin{aligned} a_0 &= \frac{2}{2\pi} \int_0^{\pi} x \, dx \, = \, \frac{\pi}{2} \\ a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos\left(n\,x\right) dx \, = \, \frac{2}{\pi} \Big[ \frac{1}{n} x \sin\left(n\,x\right) \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin\left(n\,x\right) dx \Big] \\ &= \, \frac{2}{\pi} \Big[ 0 - \frac{1}{n} \left( \frac{-1}{n} \cos\left(n\,x\right) \Big|_0^{\pi} \right] \, = \, \frac{2}{\pi n^2} \left( \cos\left(n\,\pi\right) - 1 \right) \, = \, \frac{-2}{\pi n^2} \left( 1 - \cos\left(n\,\pi\right) \right) \\ &= \, \begin{cases} \frac{-4}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{aligned}$$

And the Fourier series is:

f (x) = 
$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,odd}^{\infty} \frac{\cos(nx)}{n^2}$$

Verifying :

 $g1 = Plot[\pi/2 - (4/\pi) Sum[Cos[n x]/n^2, \{n, 1, 31, 2\}], \{x, -\pi, \pi\}];$ g2 = Plot[Abs[x], {x, -\pi, \pi}]; Show[g1, g2, PlotRange \rightarrow All]



This function is neither odd nor even,

so we cannot use symmetry arguments to evaluate integrals. The fourier coefficients are :

$$a_{0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{0}^{\pi} \sin x dx = \frac{1}{2\pi} (-\cos x \Big|_{0}^{\pi} \Big) = \frac{1}{\pi}$$
$$a_{n} = \frac{1}{\pi} \int_{0}^{\pi} \sin x \cos (n x) dx$$
$$b_{n} = \frac{1}{\pi} \int \sin x \sin (n x) dx$$

We evaluated integrals of these forms on the last homework assignment. The integral becomes

$$a_{n} = \frac{1}{\pi} \int_{0}^{\pi} \sin(mx) \cos(nx) dx = \frac{1}{\pi} \frac{\left[-m\cos(mx)\cos(nx) + n\sin(mx)\sin(nx)\right]}{m^{2} - n^{2}} \Big|_{0}^{\pi}$$

In this case, m=1, so:

$$a_{n} = \frac{1}{\pi} \left[ \frac{\cos x \cos (n x)}{n^{2} - 1} \right]_{0}^{\pi} = \frac{1}{\pi (n^{2} - 1)} \left( (-1) (-1)^{n} - 1 \right) = \begin{cases} 0, & n \text{ odd} \\ \frac{-2}{\pi (n^{2} - 1)}, & n \text{ even} \end{cases}$$

For the b coefficients, we have :

$$b_{n} = \frac{1}{\pi} \int_{0}^{\pi} \sin x \sin (n x) dx = \frac{n}{\pi (n^{2} - 1)} \cos (n x) \sin x \Big|_{0}^{\pi}$$

Having been trained to notice that  $\sin \pi = \sin 0 = 0$ , it is easy to see why so many students instinctively set these coefficients to zero. In fact, all the b coefficients will be zero except when n = 1. When n=1, both numerator and denominator go to zero, so one could compute  $b_1$  either by L'Hopital's rule or by setting n = 1 in the integral and evaluating:

$$b_1 = \frac{1}{\pi} \int_0^{\pi} \sin^2 x \, dx = \frac{1}{2}$$

Thus, the total Fourier series is :

f (x) = 
$$\frac{1}{\pi} - \frac{2}{\pi} \sum_{\text{even}}^{\infty} \frac{\cos(nx)}{n^2 - 1} + \frac{1}{2} \sin x$$

Verifying :

 $g1 = Plot[1 / \pi - Sum[(2 / \pi) Cos[n x] / (n^2 - 1), \{n, 2, 22, 2\}] + Sin[x] / 2, \{x, -\pi, \pi\}];$   $g2 = Plot[Sin[x], \{x, 0, \pi\}, PlotStyle \rightarrow \{Dashed, Red\}];$ Show[g1, g2]



We can see that the function is neither even nor odd. We can either integrate this function directly or recognize from the graph that its average value is zero to determine that  $a_0 = 0$ . We find  $a_n$  from:

$$a_{n} = \frac{1}{\pi} \left[ \int_{0}^{\pi/2} -\cos(nx) \, dx + \int_{\pi/2}^{\pi} \cos(nx) \, dx \right]$$
  
=  $\frac{1}{\pi n} \left[ -\sin(nx) \Big|_{0}^{\pi/2} + \sin(nx) \Big|_{\pi/2}^{\pi} \right] = \frac{1}{\pi n} \left[ -2\sin(n\pi/2) \right] = \begin{cases} 0, & \text{neven} \\ -2/\pi n, & n=1, 5, 9, ... \\ 2/\pi n, & n=3, 7, 11, ... \end{cases}$ 

Finally, we compute :

$$b_{n} = \frac{1}{\pi} \left[ \int_{0}^{\pi/2} -\sin(nx) \, dx + \int_{\pi/2}^{\pi} \sin(nx) \, dx \right] \cos(n\pi/2)$$
  
=  $\frac{1}{\pi n} \left[ \cos(nx) \Big|_{0}^{\pi/2} - \cos(nx) \Big|_{\pi/2}^{\pi} \right] =$   
 $\frac{1}{\pi n} \left[ (\cos(n\pi/2) - 1) - (\cos(n\pi) - \cos(n\pi/2)) \right] =$   
=  $\frac{1}{\pi n} \left[ 2\cos(n\pi/2) - 1 - (-1)^{n} \right]$ 

When n is odd, the  $cos(n \pi/2)$  term is zero and the other terms sum to zero, so b is zero for odd n. When n = 2, 6, 10, ...(Mod[n,4]=2), the coefficients become:

$$b_n = \frac{-4}{\pi n}$$

When n = 4, 8, 12, the coefficients are zero again, so our sin series consists only of the n = 2, 6, 10,... terms:

$$= \frac{-4}{\pi} \sum_{n=2,6,10}^{\infty} \sin(nx)/n$$

Verifying :

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Plot[-2 Sum[Sin[n \pi/2] Cos[n x]/(\pi n), {n, 1, 61}] -
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(4/\pi) Sum[Sin[n x]/n, {n, 2, 52, 4}], {x, -\pi, \pi}]
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3. We work with the function :



We can see from the graph that this is an even function which allows to set all  $b_n = 0$  via symmetry. We can either do the trivial integral to determine  $a_0$  or realize that the function has a value of 1 over 1/4 of its length and a value of 0 elsewhere, therefore the average value (and so the value of  $a_0$  is 1/4. We find the Subscript[a, n] coefficients :

$$a_n = \frac{1}{\pi} \int_{-\pi/4}^{\pi/4} \cos(nx) \, dx = \frac{2}{\pi} \int_{0}^{\pi/4} \cos(nx) = \frac{2}{\pi n} \sin(n\pi/4)$$

The reason the question asks you to compute the series out to the cos 8x term is that these coefficients will not "recycle" until we reach n = 8. Remembering that sin is positive in the first and second quadrants and negative in the third and fourth, we have:

$$n = 1 \Rightarrow \sin(\pi/4) = \sqrt{2}/2; \quad n = 5 \Rightarrow \sin(5\pi/4) = -\sqrt{2}/2$$

$$n = 2 \Rightarrow \sin(\pi/2) = 1; \quad n = 6 \Rightarrow \sin(3\pi/2) = -1$$

$$n = 3 \Rightarrow \sin(3\pi/4) = \sqrt{2}/2; \quad n = 7 \Rightarrow \sin(7\pi/2) = -\sqrt{2}/2$$

$$n = 4 \Rightarrow \sin(\pi) = 0; \quad n = 8 \Rightarrow \sin(2\pi) = 0$$

The fourier series is then simply:

$$f(x) = \frac{1}{4} + \left(\frac{2}{\pi}\right) \left[\frac{\sqrt{2}}{2}\cos x + \frac{\cos 2x}{2} + \frac{\sqrt{2}}{2}\frac{\cos(3x)}{3} - \frac{\sqrt{2}}{2}\frac{\cos 5x}{5} - \frac{\cos 6x}{6} - \frac{\sqrt{2}}{2}\frac{\cos(7x)}{7}\right]$$

b) When  $x = \pi/4$ , the cos 2 x and cos 6 x terms go to zero; the remaining terms all have absolute value  $\sqrt{2}/2$ , and are positive in the first and fourth quadrants and negative in the second and third, we get:

f (x) = 
$$\frac{1}{4} + \frac{2}{\pi} \cdot \frac{1}{2} \left[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... \right]$$

Thus, writing this in the form of a + b S, a = 1/4,  $b = 1/\pi$ , and the series is the expression in brackets.

c) Let me do this a little differently from the book. We know from the Dirichlet theorem that the fourier series will converge to the function where the function is continuous, and will converge to the midpoint of a discontinuity. For this function,  $x = \pi/4$  is a discontinuity, so the value of f (x) at  $x = \pi/4$  is the midpoint of the discontinuity, or 1/2. This means that :

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{\pi} \Big[ 1 - \frac{1}{3} + \frac{1}{5} - \dots \Big]$$

This series, first published around 1670 is called the Gregory series. In those times, before any sort of automatic computing, these expressions were used to solve for the value of  $\pi$ . We can rewrite the series above as:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

Unfortunately, this is a very slowly converging series, requiring over a thousand terms to get 3 decimal accuracy in the evaluation of  $\pi$ . We can verify our equation if we sum :

$$\operatorname{Sum}\left[\left(-1\right)^{n}\left(n+1\right)/\left(2n-1\right), \left\{n, 1, \infty\right\}\right]$$

Or if we compute the numerical value of the series we get:

$$Sum[(-1)^{(n+1)} / (2n-1), \{n, 1, \infty\}] / / N$$
  
0.785398

We can write a short *Mathematica* program to compute the nth partial sum and observe the rate of convergence. In the program below, nterms represents the number of terms in the partial sum:



The horizontal line is the known value of  $\pi$ ; can you figure out why the plot of partial sums has two branches? I have printed out the value of the partial sum of the first 1000 terms; compare this to the known value of  $\pi = 3.14159...$ 

4. In the program below, money[n] represents the amount of money remaining after n months. Our initial value will be money[0] = 1000



Money remaining after 3 years = 696.413 dollars

In the graph above, money remaining is on the vertical axis and time (in months) is on the horizontal.