PHYS 301
HOMEWORK #5

Solutions

1. If we make the odd extension for this function, we will have an odd function on the interval [-L, L] . This extended function has a periodicity of 2 L, and so we can find the Fourier series for the function on [-L, L], and use only the portion of that series that applies to the actual physical object (which lies on [0, L]). Since we have an odd function on our interval, we know that all the a_n coefficients are zero, and that we can write the b_n coefficients as :

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin(n\pi x/L)\, dx = \frac{2}{L}\int_{0}^{L} f(x)\sin(n\pi x/L)\, dx$$

We now need to express our function representing the string. The function consists of two straight lines between x = 0 and x = L/2, and the function is 0 between L/2 and L. We can express all this as :

$$f(x) = \begin{cases} 4hx/L, & 0 < x < L/4 \\ 2h - 4hx/L, & L/4 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

and find our integrals from the somewhat messy :

```
Clear[h, L, x]
Simplify[(2/L) (Integrate[(4 h x / L) Sin[n π x / L], {x, 0, L / 4}] + Integrate[
    (2 h - 4 h x / L) Sin[n π x / L], {x, L / 4, L/2}]), Assumptions → n ∈ Integers]
```

$$\frac{64\, h \operatorname{Cos}\left[\frac{n\pi}{8}\right] \operatorname{Sin}\left[\frac{n\pi}{8}\right]^3}{n^2\pi^2}$$

While this is a more complex form of Fourier coefficient than we are used to looking at, we can make some quick observations about the nature of the coefficients. First, we notice that the coefficients vary as the inverse square of n, suggesting this series will converge fairly rapidly. Second, we notice that the presence of a cos ($n\pi/8$) term will cause the coefficients to go to zero whenever n = 4 Mod 8. The presence of the sin term allows us to conclude that the coefficients will go to zero whenever n = 8 Mod 8. We can compute the coefficients by amending our program above :

```
Clear[b, h, L, x]
b[n_] := Simplify[(2/L) (Integrate[(4 h x / L) Sin[n π x / L], {x, 0, L / 4}] + Integrate[
      (2 h - 4 h x / L) Sin[n π x / L], {x, L / 4, L/2}]), Assumptions → n ∈ Integers]
Do[Print["n = ", n, " ⇒ b[n] = ", b[n]], {n, 8}]
```

$n = 1 \Rightarrow b[n] = \frac{8\left(-1+\sqrt{2}\right)h}{\pi^2}$

$n = 2 \Rightarrow b[n] = \frac{4\,h}{\pi^2}$

$n = 3 \Rightarrow b[n] = \frac{8\left(1+\sqrt{2}\right)h}{9\,\pi^2}$

$n = 4 \Rightarrow b[n] = 0$

$n = 5 \Rightarrow b[n] = -\frac{8\left(1+\sqrt{2}\right)h}{25\,\pi^2}$

$n = 6 \Rightarrow b[n] = -\frac{4\,h}{9\,\pi^2}$

$n = 7 \Rightarrow b[n] = -\frac{8\left(-1+\sqrt{2}\right)h}{49\,\pi^2}$

$n = 8 \Rightarrow b[n] = 0$

Or, if we wish to compute these by hand, we can go back to the expression we obtained from Mathematica :

$$b_n = \frac{64\,h\,\mathrm{Cos}[n\,\pi/8]\,\mathrm{Sin}[n\,\pi/8]^3}{n^2\,\pi^2}$$

and see if we can write this in a format that makes it easier to compute coefficients. The motivation for this is that most of us are much more familiar with the values of sin and cos at n $\pi/4$ than at n $\pi/8$. If we look carefully at the trig component of the expression above, we see that we can rewrite it as :

$$\mathrm{Cos}[n\,\pi/8]\,\mathrm{Sin}[n\,\pi/8]\,\mathrm{Sin}[n\,\pi/8]^2$$

We apply the double angle formula for sin to the first two terms to get :

$$\mathrm{Cos}[n\,\pi/8]\,\mathrm{Sin}[n\,\pi/8] = \frac{1}{2}\,\mathrm{Sin}[n\,\pi/4]$$

and we use the double angle formula for cos to note :

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$$

so that we can write :

$$\mathrm{Sin}[n\,\pi/8]^2 = \frac{1 - \mathrm{Cos}[n\,\pi/4]}{2}$$

Putting all these back into our expression for the b coefficient we have :

$$b_n = \frac{64\,h\,\frac{\text{Sin}[n\,\pi/4]}{2}\,\frac{(1-\text{Cos}[n\,\pi/4])}{2}}{n^2\,\pi^2} = \frac{16\,h\,\text{Sin}[n\,\pi\,/\,4]\,(1-\text{Cos}[n\,\pi\,/\,4])}{n^2\,\pi^2}$$

I don' t know why this isn' t the form Mathematica outputs. Now, let' s show that this expression yields the proper coefficient for the case when n = 1; the same analsysis will provide all the other coefficients. When n = 1 we get :

$$b_n = \frac{16\,h\,\text{Sin}[\pi\,/\,4]\,(1-\text{Cos}[\pi\,/\,4]}{\pi^2} = \frac{16\,h\left(\frac{\sqrt{2}}{2}\right)\left(1-\frac{\sqrt{2}}{2}\right)}{\pi^2} = \frac{4\,h\,\sqrt{2}\,\left(2-\sqrt{2}\right)}{\pi^2}$$

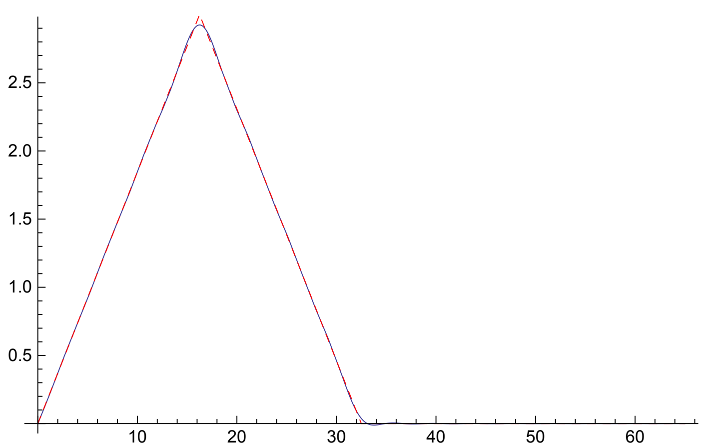
$$= \frac{8\,h\left(\sqrt{2}-1\right)}{\pi^2}$$

Then the Fourier series for this function is :

$$f\,(x) = \frac{8\,h}{\pi^2}\Big[\left(\sqrt{2}-1\right)\text{Sin}[\pi\,x\,/\,L] + \frac{\text{Sin}[\,2\,\pi\,x\,/\,L]}{2} + \frac{\left(1+\sqrt{2}\right)\text{Sin}[3\,\pi\,x\,/\,L]}{9}$$
$$- \frac{\left(1+\sqrt{2}\right)\text{Sin}[5\,\pi\,x\,/\,L]}{25} - \frac{\text{Sin}[\,6\,\pi\,x\,/\,L]}{18} - \ldots\Big]$$

Verifying that this series converges to our initial function (Remember that we need to include explicit values for L and h to produce a plot) :

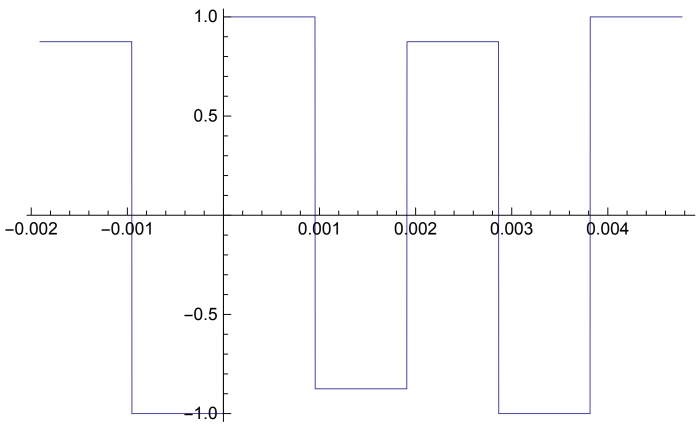
```
Clear[f, b, h, L]
h = 3; L = 65;
b[n_] := 64 h Cos[n π / 8] Sin[n π / 8]^3 / (n π)^2
g1 = Plot[Sum[b[n] Sin[n π x / L], {n, 1, 31}], {x, 0, L}];
f[x_] := Which[0 < x < L / 4, 4 h x / L, L / 4 < x < L / 2, 2 h - 4 h x / L, L / 2 < x < L, 0];
g2 = Plot[f[x], {x, 0, L}, PlotStyle → {Red, Dashed}];
Show[g1, g2]
```



And they match.

2. Let' s start plotting :

```
Clear[p, t]
p[t_] := Which[-1/524 < t < -1/1048, 7/8,
  -1/1048 < t < 0, -1, 0 < t < 1/1048, 1, 1/1048 < t < 1/524, -7/8,
  1/524 < t < 3/1048, 7/8, 3/1048 < t < 1/262, -1, 1/262 < t < 5/1048, 1]
Plot[p[t], {t, -1/524, 5/1048}]
```



And we can see that this is an odd periodic function of period 1/262 seconds, so that in our computations we will set L = 1/524. Also, since this is a pressure disturbance that repeats over and over again, we do not need to make any extension since the given function is already 2 L periodic and satisfies Dirichlet' s conditions for finding a Fourier series.

Since this is an odd function, we need only compute the b_n coefficients; making use of symmetry we have :

$$b_n = \frac{2}{L}\int_0^L p(t)\,\text{Sin}[\,n\,\pi\,t\,/\,L]\,dt$$

and again remember that here, L = 1/524 s.

Our coefficients are then :

$$b_n = \frac{2}{1/524}\Big[\int_0^{1/1048} 1\cdot \text{Sin}[n\,\pi\,t\,/\,(1\,/\,524)]\,dt - \frac{7}{8}\int_{1/1048}^{1/524}\text{Sin}[n\,\pi\,t\,/\,(1\,/\,524)\,dt]$$

This may look like an awkward integral, but it works out fairly simply :

$$b_n = 1048\Big[\frac{-1}{524\,n\,\pi}\Big((\cos(n\,\pi/2) - 1) - \frac{7}{8}(\cos(n\,\pi) - \cos(n\,\pi/2))\Big)\Big]$$

$$= \frac{-2}{n\,\pi}\Big[\frac{15}{8}\cos(n\,\pi/2) - 1 - \frac{7}{8}(-1)^n\Big] = \frac{1}{4\,n\,\pi}[8 + 7\,(-1)^n - 15\cos(n\,\pi/2)]$$

Let' s look at the final expression for b_n to see what patterns we can find. When n is odd, the cos(n π/2) term will be zero, and the sum inside the bracket will always be 1, so that all odd terms will have the value 1/(4 n π). When n = 4, 8, 12, ..., the cos(n π/2) term will be plus 1, and the sum inside the brackets will be zero, so every fourth coefficient will be zero. When n = 2, 6, 10, ... the cos(n π/2) term will be - 1, and the sum inside the brackets will be 30, so that these coefficients will have a value of 15/(2 n π). We can summarize these results in this table :

n	$\mathbf{b_n}$
1	$1/4\pi$
2	$15/4\pi$
3	$1/12\pi$
4	0
5	$1/20\pi$
6	$5/4\pi$
7	$1/28\pi$
8	0

The Fourier series is then :

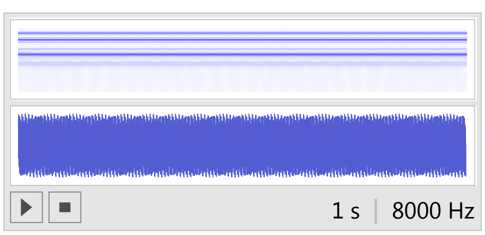
$$p(t) = \frac{1}{4\pi}\left[\mathrm{Sin}[524\,\pi\,t] + 15\,\mathrm{Sin}[2\cdot 524\,\pi\,t] + \frac{\mathrm{Sin}[3\cdot 524\,\pi\,t]}{3} + \frac{\mathrm{Sin}[5\cdot 524\,\pi\,t]}{5} + 5\,\mathrm{Sin}[6\cdot 524\,\pi\,t] + \frac{\mathrm{Sin}[7\cdot 524\,\pi\,t]}{7} + \ldots\right]$$

Note that the first term (of frequency 262 Hz) does not have the largest amplitude; the second term (frequency = 524 Hz) has an amplitude 15 times greater than the n = 1 term, meaning that the second term contains 225 times more energy than the first. The n = 6 term also contributes more to the overall wave pattern than the n = 1 term.

You can listen to the sound of this wave pattern by typing in the code below and entering; when you execute the code you will get the panel that you see in the output statement. Click the play button and you will hear the sound of this wave package. You can adjust the number of terms in the sum to see what it sounds like with more or fewer harmonics; remember, that the higher harmonics will have very small amplitudes.

In[195]:=
```
Clear[b]
b[n_] := (1/4 n π) (8 + 7 (-1)^n - 15 Cos[n π/2])
Sound[Play[Sum[b[n] Sin[524 n π t], {n, 1, 11}], {t, 0, 1}]]
```

Out[197]=



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3. Since this waveform repeats 60 cycles/s, the function defined on [0, 1/60 s] is a 2 L periodic function for which we can determine a Fourier series. Here, 2 L = 1/60 s so that L = 1/120 s. The current increases from 0 to 10 A in 1/120 s, so the straight line representing this increase is simply :

$$i(t) = 1200\,t, \;\; 0 < t < 1/120\,\text{s}$$

Now we compute Fourier coefficients:

We can find a_0 either by integrating or by inspection: the average of the current in the first half of the cycle is 5 amps, and is zero in the second half of the cycle. Thus the average over one cycle is 5/2 amps.

$$a_n = 120 \cdot 1200 \int_0^{1/120} t \cos(120\, n\, \pi\, t)\, dt$$

$$= 120 \cdot 1200 \left[\frac{1}{120\, n\, \pi} t \sin(120\, n\, \pi\, t) \Big|_0^{1/120} - \frac{1}{120\, n\, \pi} \int_0^{1/120} \sin(120\, n\, \pi\, t)\, dt \right]$$

Since $\sin(n\,\pi) = \sin 0 = 0$, the first term in brackets is zero. We have then:

$$a_n = \frac{-120 \cdot 1200}{120\, n\, \pi} \int_0^{1/120} \sin(120\, n\, \pi\, t)\, dt = \frac{120 \cdot 1200}{(120\, n\, \pi)^2} \cos(120\, n\, \pi\, t) \Big|_0^{1/120}$$

$$= \frac{10}{n^2\, \pi^2} (\cos(n\, \pi) - 1) = \begin{cases} -20 \big/ n^2\, \pi^2, & \text{n odd} \\ 0, & \text{n even} \end{cases}$$

$$b_n = \frac{1}{1/120} \int_0^{1/120} 1200\, t \sin(120\, n\, \pi\, t)\, dt$$

$$= 1200 \cdot 120 \left[\frac{-1}{120\, n\, \pi} t \cos(120\, n\, \pi\, t) \Big|_0^{1/120} + \frac{1}{120\, n\, \pi} \int_0^{1/120} \cos(120\, n\, \pi\, t)\, dt \right]$$

It should be easy to show that the final integral on the right is zero, leaving us with:

$$b_n = -\frac{1200 \cdot 120}{120\, n\, \pi} \left(\frac{1}{120} \cos(n\, \pi) \right) = \frac{-10\,(-1)^n}{n\, \pi}$$

Our Fourier series is:

$$i(t) = \frac{5}{2} - \frac{20}{\pi^2} \left[\text{Cos}[120\, \pi\, t] + \frac{\text{Cos}[3 \cdot 120\, \pi\, t]}{9} + \frac{\text{Cos}[5 \cdot 120\, \pi\, t]}{25} + \ldots \right]$$
$$+ \frac{10}{\pi} \left[\text{Sin}[120\, \pi\, t] - \frac{\text{Sin}[2 \cdot 120\, \pi\, t]}{2} + \frac{\text{Sin}[3 \cdot 120\, \pi\, t]}{3} - \ldots \right]$$

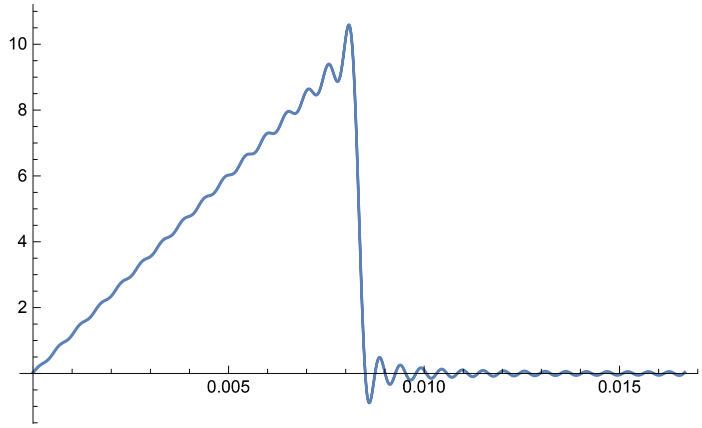
We can write this in closed form as :

$$i(t) = \frac{5}{2} - \frac{20}{\pi^2} \sum_{\text{odd}}^{\infty} \frac{\text{Cos}[120\, n\, \pi\, t]}{n^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\text{Sin}[120\, n\, \pi\, t]}{n}$$

Verifying with *Mathematica*:

In[234]:=
```
Plot[5/2 - (20/π^2) Sum[Cos[120 n π t]/n^2, {n, 1, 31, 2}] -
  (10/π) Sum[(-1)^n Sin[120 n π t]/n, {n, 1, 31}], {t, 0, 1/60}]
```

Out[234]=



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4. We have an even 2 L periodic function defined on [-1/2, 1/2]. This implies L = 1/1 and we write :

$$a_0 = \frac{1}{2\,L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx = 2\int_0^{1/2} x^2\,dx = \frac{1}{12}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(n\,\pi\,x/L)\,dx = 4\int_0^{1/2} x^2\cos(2\,n\,\pi\,x)\,dx$$

Simplify[4 Integrate[x^2 Cos[2 n π x], {x, 0, 1/2}], Assumptions → n ∈ Integers]

$$\frac{(-1)^n}{n^2\,\pi^2}$$

The b_n coefficients are all zero since f(x) is even. Our Fourier series becomes:

$$f(x) = \frac{1}{12} - \frac{1}{\pi^2}\left[\mathrm{Cos}[2\,\pi\,x] - \frac{\mathrm{Cos}[4\,\pi\,x]}{4} + \frac{\mathrm{Cos}[6\,\pi\,x]}{9} - \frac{\mathrm{Cos}[8\,\pi\,x]}{16} + \frac{\mathrm{Cos}[10\,\pi\,x]}{25} - \ldots\right]$$

and can be written in closed form as:

$$f(x) = \frac{1}{12} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\,\mathrm{Cos}[2\,n\,\pi\,x]}{n^2}$$

```
g1 = Plot[1/12 + (1/π^2) Sum[(-1)^n Cos[2 n π x]/n^2, {n, 1, 21}], {x, -3/2, 3/2}];
g2 = Plot[x^2, {x, -1/2, 1/2}, PlotStyle → {Dashed, Red}];
Show[g1, g2]
```