# PHYS 301 HOMEWORK \#6 

## Due : 24 Feb 2016

You may use Mathematica to compute Fourier coefficients. If you do, please include the relevant Mathematica output with your solution.

1. Problem 9-60 from text
a) $e^{i n \pi}=\cos (\mathrm{n} \pi)+\mathrm{i} \sin (\mathrm{n} \pi)=(-1)^{n}$
b) $\mathrm{e}^{\mathrm{in} \pi}-\mathrm{e}^{-\mathrm{in} \pi}=\cos (\mathrm{n} \pi)+\mathrm{i} \sin (\mathrm{n} \pi)-(\cos (\mathrm{n} \pi)-\mathrm{i} \sin (\mathrm{n} \pi))=2 \mathrm{i} \sin (\mathrm{n} \pi)=0$
c) $\mathrm{e}^{2 \mathrm{in} \pi}=\cos (2 \mathrm{n} \pi)+\mathrm{i} \sin (2 \mathrm{n} \pi)=\cos ^{2}(2 \mathrm{n} \pi)-\sin ^{2}(2 \mathrm{n} \pi)+\mathrm{i} \sin (2 \mathrm{n} \pi)$

$$
=1
$$

since $\cos (2 \mathrm{n} \pi)=1$ and $\sin (2 \mathrm{n} \pi)=0$ for all integers $n$.
2. Find the complex Fourier series for the function $f(x)=x$ on $[-\pi, \pi]$. Find an expression for the complex coefficients, and show that the complex Fourier series matches the Fourier trig series out to the $e^{ \pm 5 i x}$ terms. (Assume the function repeats so that the Dirichlet conditions are satisfied).
Solution: Since the average value of the function on the interval is zero, we can conclude that the value of $c_{0}=0$. Alternately, we could integrate :

$$
c_{0}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} x d x=0
$$

The $c_{n}$ coefficients are found from:

$$
\mathrm{c}_{\mathrm{n}}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \mathrm{xe}^{-\mathrm{inx}} \mathrm{dx}=\frac{-(-1)^{\mathrm{n}}}{\mathrm{in}}
$$

as shown by Mathematica:

$$
\begin{aligned}
& \text { Simplify[Integrate[x } \operatorname{Exp}[-i \mathbf{n x}],\{\mathbf{x},-\pi, \pi\}] /(2 \pi), \text { Assumptions } \rightarrow \mathbf{n} \in \text { Integers }] \\
& \frac{i(-1)^{\mathrm{n}}}{\mathrm{n}}
\end{aligned}
$$

and remembering that $\mathrm{i}=-1 / \mathrm{i}$. Since $c_{0}=0$ we can write our terms:

$$
f(x)=c_{1} e^{i x}+c_{-1} e^{-i x}+c_{2} e^{2 i x}+c_{-2} e^{-2 i x}+\ldots
$$

$$
f(x)=\left(\frac{e^{i x}}{i}-\frac{e^{-i x}}{i}\right)+\left(\frac{-e^{2 i x}+e^{-2 i x}}{2 i}\right)+\left(\frac{e^{3 i x}-e^{-3 i x}}{3 i}\right)+\left(\frac{-e^{4 i x}-e^{-4 i x}}{4 i}\right)+\left(\frac{e^{5 i x}-e^{-5 i x}}{5 i}\right)+\ldots
$$

In determining this expression, remember that the sign of the coefficient alternates, and also to take into account the sign of n when computing the $c_{-n}$ terms. Studying the Fourier series above, we recognize that the exponentials can be grouped as:

$$
f(x)=2\left[\sin x-\frac{\sin 2 x}{2}+\frac{\sin 3 x}{3}-\frac{\sin 4 x}{4}+\frac{\sin 5 x}{5}-\ldots\right]
$$

remembering the definition of $\sin \mathrm{nx}=\left(e^{i n x}-e^{-i n x}\right) / 2 \mathrm{i}$. The expression in eq. (1) matches the Fourier series from HW 4 problem 2a.
3. Find the complex Fourier series for the function :

$$
\mathrm{f}(\mathrm{x})= \begin{cases}0, & -3<\mathrm{x}<\mathrm{L} \\ \mathrm{x}, & 0<\mathrm{x}<\mathrm{L}\end{cases}
$$

Assume the function repeats so that the Dirichlet conditions are satisfied. Find an expression for the complex Fourier coefficients and show that the complex series matches the trig series out to the $\mathrm{n}=$ 5 terms.

Solution: The function as corrected in email is :

$$
\mathrm{f}(\mathrm{x})= \begin{cases}0, & -3<\mathrm{x}<0 \\ \mathrm{x}, & 0<\mathrm{x}<3\end{cases}
$$

We wish to compute a Fourier series of the form:

$$
\mathrm{f}(\mathrm{x})=\sum \underset{\substack{\mathrm{n} \\ \mathrm{n}=-\infty}}{\substack{\mathrm{i} \mathrm{inx} \mathrm{x} / 3}}
$$

with :

$$
\begin{gathered}
c_{0}=\frac{1}{6} \int_{0}^{3} x d x=\frac{3}{4} \\
c_{n}=\frac{1}{6} \int_{0}^{3} x^{-i n \pi x / 3} d x
\end{gathered}
$$

Which we can compute via:

```
Simplify[Integrate[x Exp[-ín \pix/3], {x,0,3}]/6, Assumptions }->\textrm{n}\in\mathrm{ Integers]
- (-1\mp@subsup{)}{}{n}(-1+(-1\mp@subsup{)}{}{n}-\mathrm{ i }n\pi)
```

Let' s separate this result into a real and an imaginary part; we know the real part will give us cos terms, and the imaginary part will yield sin terms :

$$
\mathrm{c}_{\mathrm{n}}=\frac{-3(-1)^{\mathrm{n}}\left(-1+(-1)^{\mathrm{n}}\right)}{2 \mathrm{n}^{2} \pi^{2}}+\mathrm{i} \frac{3(-1)^{\mathrm{n}}}{2 \mathrm{n} \pi}
$$

We can evaluate the real part :

$$
\operatorname{Re}\left(\mathrm{c}_{\mathrm{n}}\right)= \begin{cases}0, & \mathrm{n} \text { even } \\ -3 / \mathrm{n}^{2} \pi^{2}, & \mathrm{n} \text { odd }\end{cases}
$$

Note that because of the $n^{2}$ term, the $c_{n}$ and $c_{-n}$ terms will have the same sign.

The imaginary part is:

$$
\operatorname{Im}\left(c_{n}\right)= \begin{cases}3 /(2 \mathrm{in} \pi), & \mathrm{n} \text { odd } \\ -3 /(2 \mathrm{in} \pi), & \mathrm{n} \text { even }\end{cases}
$$

Remember that $1 / \mathrm{i}=-\mathrm{i}$. Therefore, when we write our the Fourier series we will have both odd and even terms with an i in the denominator; these will yield sin terms. There will be only odd real coefficients, so there will be only odd cos terms. Writing out our Fourier series:

$$
\begin{gathered}
\mathrm{f}(\mathrm{x})=\frac{3}{4}+\frac{3}{\pi}\left[\frac{\mathrm{e}^{\mathrm{i} \pi \mathrm{x} / 3}-\mathrm{e}^{-\mathrm{i} \pi \mathrm{x} / 3}}{2 \mathrm{i}}-\left(\frac{\mathrm{e}^{2 \mathrm{i} \pi \mathrm{x} / 3}-\mathrm{e}^{-2 \mathrm{i} \pi \mathrm{x} / 3}}{2(2 \mathrm{i})}\right)+\right. \\
\left.\left(\frac{\mathrm{e}^{3 \mathrm{i} \pi \mathrm{x} / 3}-\mathrm{e}^{-3 \mathrm{i} \pi \mathrm{x} / 3}}{3(2 \mathrm{i})}\right)-\left(\frac{\mathrm{e}^{4 \mathrm{i} \pi \mathrm{x} / 3}-\mathrm{e}^{-4 \mathrm{i} \pi \mathrm{x} / 3}}{4(2 \mathrm{i})}\right)+\left(\frac{\mathrm{e}^{5 \mathrm{i} \pi \mathrm{x} / 3}-\mathrm{e}^{-5 \mathrm{i} \pi \mathrm{x} / 3}}{5(2 \mathrm{i})}\right)\right] \\
\frac{-3}{\pi^{2}}\left[\left(\mathrm{e}^{\mathrm{i} \pi \mathrm{x} / 3}+\mathrm{e}^{-\mathrm{i} \pi \mathrm{x} / 3}\right)+\left(\frac{\mathrm{e}^{3 \mathrm{i} \pi \mathrm{x} / 3}+\mathrm{e}^{-3 \mathrm{i} \pi \mathrm{x} / 3}}{9}\right)+\left(\frac{\mathrm{e}^{5 \mathrm{i} \pi \mathrm{x} / 3}+\mathrm{e}^{-5 \mathrm{i} \pi \mathrm{x} / 3}}{25}\right)+\ldots\right]
\end{gathered}
$$

Writing this in terms of trig functions:

$$
\begin{aligned}
\mathrm{f}(\mathrm{x})= & \frac{3}{4}+\frac{3}{\pi}\left[\operatorname{Sin}(\pi \mathrm{x} / 3)-\frac{\operatorname{Sin}(2 \pi \mathrm{x} / 3)}{2}+\frac{\operatorname{Sin}(3 \pi \mathrm{x} / 3)}{3}-\ldots\right]- \\
& \frac{6}{\pi^{2}}\left[\operatorname{Cos}(\pi \mathrm{x} / 3)+\frac{\operatorname{Cos}(3 \pi \mathrm{x} / 3)}{9}+\frac{\operatorname{Cos}(5 \pi \mathrm{x} / 3)}{25}+\ldots\right]
\end{aligned}
$$

Compare this to the trig coefficients:

```
a0 = Integrate [x,{x, 0, 3}]/6
3
an}=\mathrm{ Simplify[Integrate[x Cos[n mx/3],{x, 0, 3}]/3, Assumptions }->\textrm{n}\in\mathrm{ Integers]
3(-1+(-1\mp@subsup{)}{}{n})
b
- 午(-1)n
```

Use these coefficients to produce a trig series and we find:

$$
\mathrm{f}(\mathrm{x})=\frac{3}{4}-\frac{6}{\pi^{2}} \sum_{\text {odd }}^{\infty} \operatorname{Cos}[\mathrm{n} \pi \mathrm{x} / 3] / \mathrm{n}^{2}+\frac{3}{\pi}\left[\sum_{\mathrm{n}=1}^{\infty}(-1)^{\mathrm{n}+1} \operatorname{Sin}[\mathrm{n} \pi \mathrm{x} / 3] / \mathrm{n}\right]
$$


$\left.(3 / \pi) \operatorname{Sum}\left[(-1)^{(n+1)} \operatorname{Sin}[n \pi x / 3] / n,\{n, 1,31\}\right],\{x,-3,3\}\right]$
4. Use the function described in question 4 of HW 5 in conjunction with Parseval' s theorem to evaluate:

$$
\sum_{n=1}^{\infty} \frac{1}{n^{4}}
$$

Solution: In the last homework we computed the values of $a_{n}$ for this function as:
$a_{n}=\frac{(-1)^{n}}{n^{2} \pi^{2}}$
and $a_{0}=\frac{1}{12}$.
Since the function is even on $[-1 / 2,1 / 2]$, we know that $b_{n}=0$. To find the average value of the square of the function, we integrate:

$$
|f(x)|_{\text {avg }}^{2}=\frac{1}{1} \int_{-1 / 2}^{1 / 2} x^{4} d x=2 \int_{0}^{1 / 2} x^{4} d x=\frac{1}{80}
$$

(Note that the interval has length 1, which is reflected in the denominator of the first step of integration.)

Now, apply Parseval's Theorem:

$$
\begin{aligned}
& \mid \mathrm{f}(\mathrm{x}) \text { lavg }=a_{0}^{2}+\frac{1}{2} \Sigma a_{n}^{2}+\frac{1}{2} \Sigma \mathrm{~b}_{\mathrm{n}}^{2} \\
& \qquad \begin{array}{l}
\frac{1}{80}=\left(\frac{1}{12}\right)^{2}+\frac{1}{2} \Sigma \frac{(-1)^{2 n}}{n^{4} \pi^{4}}+0 \Rightarrow \frac{1}{80}-\frac{1}{144}=\frac{9}{720}-\frac{5}{720}=\frac{1}{180} \\
\frac{1}{180}=\frac{1}{2} \Sigma \frac{1}{\mathrm{n}^{4} \pi^{4}} \Rightarrow \Sigma \frac{1}{\mathrm{n}^{4}}=\frac{\pi^{4}}{90}
\end{array}
\end{aligned}
$$

You may use Mathematica to check your answer, but you must use Parseval's theorem to determine this result. Verifying:

```
\(\operatorname{Sum}\left[1 / n^{4},\{n, \infty\}\right]\)
\(\frac{\pi^{4}}{90}\)
```

5. Write a Mathematica program that will compute the Fourier trig coefficients for the function below; will print out the Fourier series out to the $\mathrm{n}=11$ terms, and will verify the accuracy of the series by plotting the series and the original function on the same set of axes :

$$
\mathrm{f}(\mathrm{x})= \begin{cases}0, & -2<\mathrm{x}<0 \\ \mathrm{x}^{2}, & 0<\mathrm{x}<2\end{cases}
$$

Clear[f, a0, a, b, fourier]
(* First define the function *)
$\mathrm{f}\left[\mathrm{x}_{-}\right]:=$Which $\left[-2<\mathrm{x}<0,0,0<\mathrm{x}<2, \mathrm{x}^{\wedge} 2\right]$
(* Compute Fourier coefficients *)
$a 0=$ Integrate $[f[x],\{x,-2,2\}] / 4 ;$
$a\left[n_{-}\right]:=a[n]=$ Integrate $[f[x] \operatorname{Cos}[n \pi x / 2],\{x,-2,2\}] / 2$
$b\left[n_{-}\right]:=b[n]=$ Integrate $[f[x] \operatorname{Sin}[n \pi x / 2],\{x,-2,2\}] / 2$
(* Construct the Fourier series *)
fourierseries $=a 0+\operatorname{Sum}[a[n] \operatorname{Cos}[n \pi x / 2]+b[n] \operatorname{Sin}[n \pi x / 2],\{n, 11\}]$;
(* Write the first 11 terms of the Fourier series *)

Expand[fourierseries]

```
(* Define two graphics;
one for the Fourier series and the second for the function *)
g1 = Plot[fourierseries, {x, -2, 2}];
g2 = Plot[f[x], {x, -2, 2}, PlotStyle }->\mathrm{ {Red, Dashed}];
Show[g1, g2]
```

$$
\begin{aligned}
& \frac{2}{3}-\frac{8 \operatorname{Cos}\left[\frac{\pi x}{2}\right]}{\pi^{2}}+\frac{2 \operatorname{Cos}[\pi x]}{\pi^{2}}-\frac{8 \operatorname{Cos}\left[\frac{3 \pi x}{2}\right]}{9 \pi^{2}}+\frac{\operatorname{Cos}[2 \pi x]}{2 \pi^{2}}-\frac{8 \operatorname{Cos}\left[\frac{5 \pi x}{2}\right]}{25 \pi^{2}}+ \\
& \frac{2 \operatorname{Cos}[3 \pi x]}{9 \pi^{2}}-\frac{8 \operatorname{Cos}\left[\frac{7 \pi x}{2}\right]}{49 \pi^{2}}+\frac{\cos [4 \pi x]}{8 \pi^{2}}-\frac{8 \operatorname{Cos}\left[\frac{9 \pi x}{2}\right]}{81 \pi^{2}}+\frac{2 \operatorname{Cos}[5 \pi x]}{25 \pi^{2}}-\frac{8 \operatorname{Cos}\left[\frac{11 \pi x}{2}\right]}{121 \pi^{2}}- \\
& \frac{16 \operatorname{Sin}\left[\frac{\pi x}{2}\right]}{\pi^{3}}+\frac{4 \operatorname{Sin}\left[\frac{\pi x}{2}\right]}{\pi}-\frac{2 \operatorname{Sin}[\pi x]}{\pi}-\frac{16 \operatorname{Sin}\left[\frac{3 \pi x}{2}\right]}{27 \pi^{3}}+\frac{4 \operatorname{Sin}\left[\frac{3 \pi x}{2}\right]}{3 \pi}-\frac{\operatorname{Sin}[2 \pi x]}{\pi}- \\
& \frac{16 \operatorname{Sin}\left[\frac{5 \pi x}{2}\right]}{125 \pi^{3}}+\frac{4 \operatorname{Sin}\left[\frac{5 \pi x}{2}\right]}{5 \pi}-\frac{2 \operatorname{Sin}[3 \pi x]}{3 \pi}-\frac{16 \operatorname{Sin}\left[\frac{7 \pi x}{2}\right]}{343 \pi^{3}}+\frac{4 \operatorname{Sin}\left[\frac{7 \pi x}{2}\right]}{7 \pi}-\frac{\operatorname{Sin}[4 \pi x]}{2 \pi}- \\
& \frac{16 \operatorname{Sin}\left[\frac{9 \pi x}{2}\right]}{729 \pi^{3}}+\frac{4 \operatorname{Sin}\left[\frac{9 \pi x}{2}\right]}{9 \pi}-\frac{2 \operatorname{Sin}[5 \pi x]}{5 \pi}-\frac{16 \operatorname{Sin}\left[\frac{11 \pi x}{2}\right]}{1331 \pi^{3}}+\frac{4 \operatorname{Sin}\left[\frac{11 \pi x}{2}\right]}{11 \pi}
\end{aligned}
$$

We can see how the Fourier series converges to $f(x)$, but with relatively few terms, there is still some "wobble" around the function.

