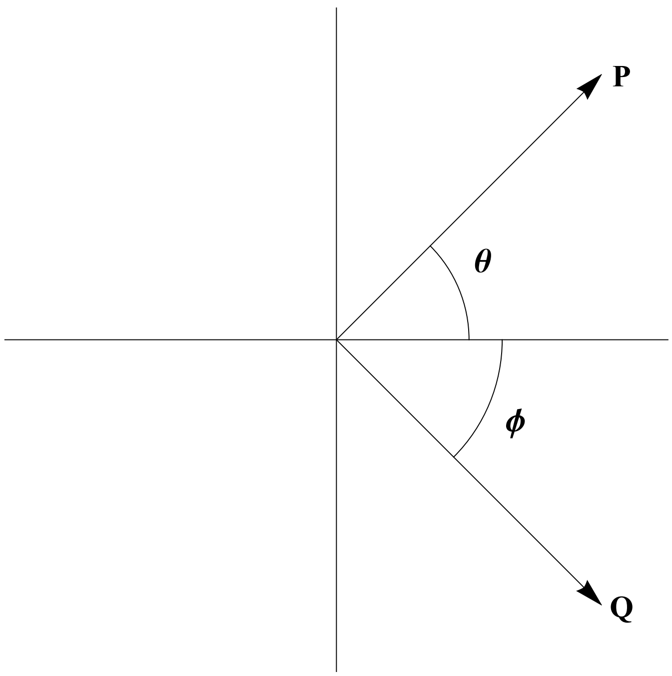
PHYS 301
HOMEWORK #7-- Solutions

1. Consider the diagram below :



The magnitude of the cross-product of P and Q can be written as:

$$\mathbf{P}\times\mathbf{Q} \;=\; |\,P\,|\,|\,Q\,|\;\sin(\theta+\phi)$$

Alternately, we can compute the cross product by using determinants:

$$\begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ P\cos\theta & P\sin\theta & 0 \\ Q\cos\phi & -Q\sin\phi & 0 \end{pmatrix}$$

The only non zero component of this determinant is the z component, and has magnitude:

$$P\cos\theta\, Q\sin\phi \;+\; P\sin\theta\, Q\cos\phi \;=\; P\,Q\,(\cos\theta\sin\phi \;+\; \sin\theta\cos\phi)$$

Since both expressions of magnitude must be equal, we can set:

$$P\,Q\sin(\theta+\phi) \;=\; P\,Q\,(\cos\theta\sin\phi \;+\; \sin\theta\cos\phi)$$

which yields the double angle formula:

$$\sin(\theta+\phi) \;=\; \sin\theta\cos\phi \;+\; \cos\theta\sin\phi$$

2. For a wind profile given by $\mathbf{v} = x^3\ \mathbf{i} + \left(x^2 + y^3\right)\ \mathbf{j}$:

a) Wind velocity at the origin is 0.

b) Wind velocity at (1, 2) is $\mathbf{i} + (1 + 8)\ \mathbf{j} = \mathbf{i} + 9\ \mathbf{j}$

c) Wind velocity at (3, 0) is $27\ \mathbf{i} + 9\ \mathbf{j}$

e) The only component of velocity that will cross the boundary at (3, 0) is the x component of velocity. In other words, the velocity component must be perpendicular to the portion of the surface it crosses.

f) The rate at which waste crossing that portion of the boundary is

$$\mathrm{v_x} * \sigma\, \mathrm{dy} \;=\; 27\ \mathrm{mi/hr}\left(10^6\ \mathrm{tbq} \big/ \mathrm{mi}^2\right)(\mathrm{dy\ mi}) \;=\; 2.7\times 10^7\ \mathrm{tbq/hr}$$

Be sure that you use proper units and can verify that this expression in fact yields a result whose units are radioactive waste/hr

g, h) Here you are being asked to find the divergence of the velocity field, which is :

$$\nabla\cdot\mathbf{v} \;=\; 3\,\mathrm{x}^2 + 3\,\mathrm{y}^2 \;=\; 3\left(\mathrm{x}^2 + \mathrm{y}^2\right) \;=\; 3\,\mathrm{r}^2$$

i) Our double integral is :

$$\int_0^{2\pi}\int_0^{\mathrm{R}} 3\,\mathrm{r}^2\,(\mathrm{r\,dr\,d}\phi) \;=\; 12\,\pi\int_0^{\mathrm{R}} \mathrm{r}^3\,\mathrm{dr} \;=\; 3\,\pi\,\mathrm{R}^4$$

Multiply this by σ to find the flow rate of waste through the city.

3. For sand depth given by $z = x^2 \sin^2 y$

a) The depth will be zero wherever x = 0 and wherever $y = n\,\pi$.

b) Because of the $\sin^2 y$ factor (and because sin is bounded by ± 1), the maximum possible depth at any value of x is x^2. If you walk along the line of, say, x= 3, the depth of the sand will vary sinusoidally reaching a maximum value of 9 and minimum value of 0 (when $y = n\,\pi$).

c) If you start at $(1, \pi/2)$, the depth of the sand is 1. If you walk in the + x direction, y does not vary so the sin factor always has a value of 1, and the total depth will vary as x^2.

d) Now, the depth will vary sinusoidally with a maximum value of 1 and minimum value of 0.

For a velocity field of x^2i + 3 j :

e) If the initial position of the leaf is (0, 2), the initial velocity of the leaf will be 0 i + 3 j; in other words, the leaf will move along the + y direction at 2 m/s (assuming the units are in m/s). The leaf will never acquire velocity in the x direction, and will have constant motion along the y axis.

f) If the leaf starts at (1, 2), its initial velocity will be i + 3 j. Its speed will be $\sqrt{10}$ m/s in the direction of arctan(3/1). Since the leaf has motion in the x direction, the total speed will increase (and go as x^2). The y component of velocity will remain constant at 3 m/s, so the direction of motion will approach the positive x axis.

g) If the leaf starts at (-1, 2), its initial speed and direction are the same as in part f). However, as the particle approaches the y axis, its x velocity component approaches zero. The x component of the velocity will decrease, and the particle will move asymptotically toward the + y axis.

We can show this behavior by writing a short recursion program:

In[161]:=
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Clear[x, y, v, h]
x[0] = -1; y[0] = 2; h = 0.01;
x[n_] := x[n] = x[n - 1] + h x[n - 1]^2
y[n_] := y[n] = y[n - 1] + 3 h
ListPlot[Table[{x[n], y[n]}, {n, 1000}]]
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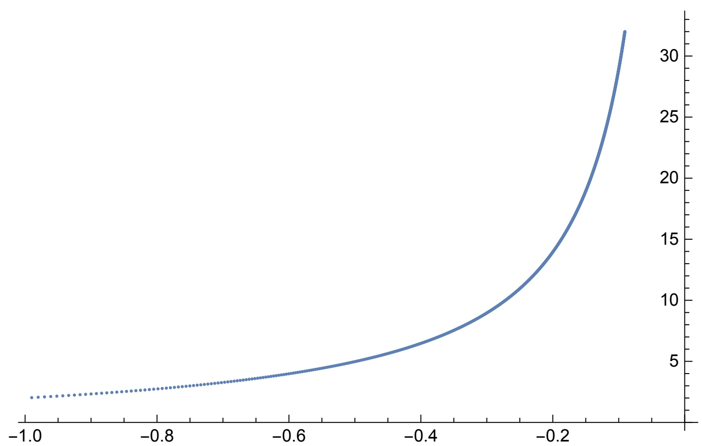
Out[165]=



In the program above, h is the step size (0.01s) between calculations. Look carefully at the equations for x[n] and y[n] . They are based on the logic:

$x(t) = x(t\text{-}\Delta t) + v_x\, \Delta t$ $\qquad\qquad$ $y(t) = y\,(t\ \text{-}\Delta t) + v_y\, \Delta t$

Here $h = \Delta t$.