PHYS 301
HOMEWORK #8-- Solutions

1. a) Ask yourself how h varies if you move a little in the plus x direction? How does h vary if you move in the + y direction? In both cases, h increases, therefore both $\partial h/\partial x$ and $\partial h/\partial y$ are positive.

b) Rocks roll downhill, that' s how gravity works by the way. On this contour plot, that means the rock would follow the steepest descent from the point "5" to the point "1".

c) Just as we have talked about in class, the greatest slope will be where the isopleths (a fancy word for lines of equal value) are most closely packed. It looks to me that this is between the "5" and "9" contours.

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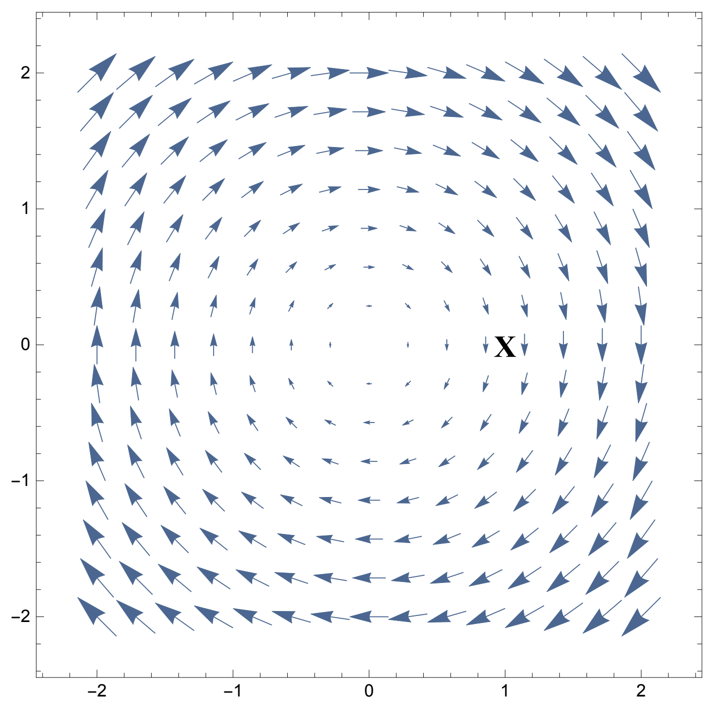
2. We' ll use the "VectorPlot" command and superimpose an **X** at the drop point :

In[51]:=
```
VectorPlot[{y, -x}, {x, -2, 2}, {y, -2, 2}, Epilog →
  Text[Style["X", FontWeight → "Bold", FontFamily → "Times", FontSize → 16], {1, 0}]]
```

Out[51]=



The particle' s initial motion will be in the negative y direction, and as you can see will execute, for all eternity, the circular path of radius 1.

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3. a) If we assume that the Earth is a point particle with all of its mass at the center, we expect that

we will run into a singularity as we approach R -> 0. Since we know that the gravitational force is :

$$g = \frac{-G\,M}{x^2}$$

where x is the distance from the center of the Earth. The potential is :

$$V = -\int g\,dx = -G\,M\,/x + C_1 \text{ (where we set the } C_1 = 0 \text{ at } x \to \infty)$$

Out plot is then :

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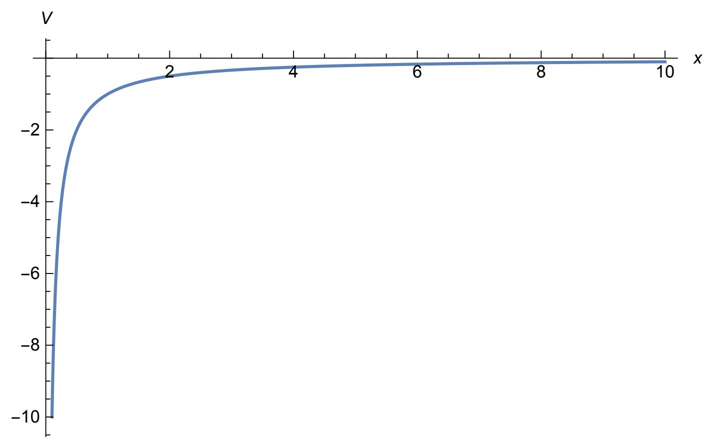
In[36]:= Plot[-1/x, {x, 0.1, 10}, PlotRange → All, AxesLabel → {x, V}]
```

Out[36]=



I am using arbitrary units where the numerator has a value of 1.

b) Assuming a spherical distribution of matter inside the Earth, a particle at the center of the Earth is equally attracted in all directions, so that the vector force sums to zero. Hence, at the center of the Earth, your weight would be zero.

c) Inside the Earth, the gravitational force varies linearly with distance from the center :

$$g = -K\,x \Rightarrow V = -\int -K\,x\,dx = \frac{K\,x^2}{2} + C \tag{1}$$

d) At the surface of the Earth (where x = R), both descriptions of the potential must be equal, so we have :

$$\frac{K\,R^2}{2} + C = \frac{-G\,M}{R} \tag{2}$$

Solving for C we obtain:

$$C = \frac{-G\,M}{R} - \frac{K\,R^2}{2}$$

e) Substituting this into equation (1):

$$V(x) = \begin{cases} -G\,M/x & x > R \\ (K/2)\left(x^2 - R^2\right) - G\,M/R, & 0 < x < R \end{cases}$$

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4. $f(x, y, z) = x \ln(y + 3z)$

$$\nabla T = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = \ln(y+3z)\,\hat{\mathbf{x}} + \frac{x}{y+3z}\hat{\mathbf{y}} + \frac{3x}{y+3z}\hat{\mathbf{z}}$$

At the point (5,7,-2):

$$\nabla T = \ln 1\,\hat{\mathbf{x}} + 5\,\hat{\mathbf{y}} + 15\,\hat{\mathbf{z}} = 5\,\hat{\mathbf{y}} + 15\,\hat{\mathbf{z}}$$

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5. Let' s start by drawing the contours for the scalar field :

In[41]:=
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ContourPlot[Exp[-x^2 - y^2], {x, -2, 2}, {y, -2, 2}, ContourLabels → All]
```
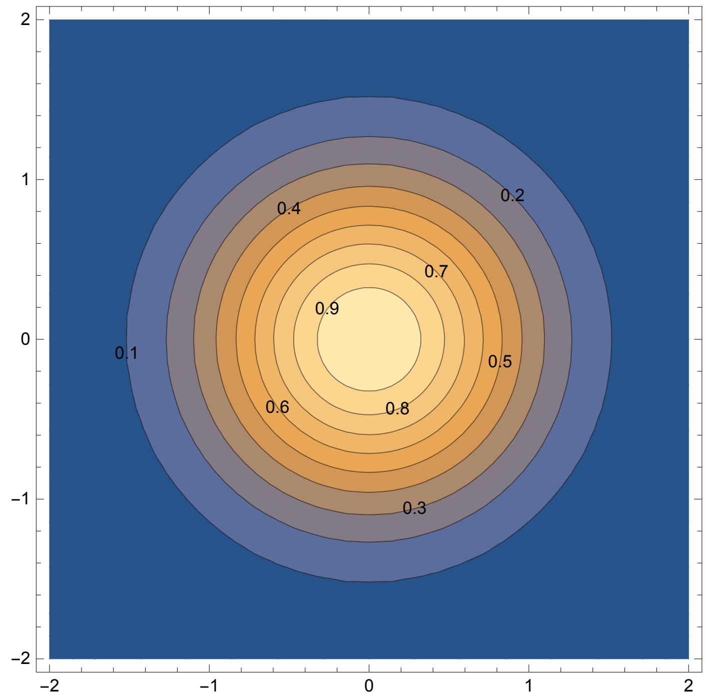

Out[41]=



a) We can see that the contour lines are concentric circles, so that the gradient, which represents the direction of greatest temperture change, will lie along a radial line toward the center (since the temperature increases inward). At the point (1, 1), this direction will be (-1, 1). We can also show this by direct computation of the gradient :

$$\nabla T = -2\,x\,e^{-(x^2+y^2)}\,\hat{\mathbf{x}} - 2\,y\,e^{-(x^2+y^2)}\,\hat{\mathbf{y}}$$

Evaluated at (1, 1) :

$$\nabla T = -2\,e^{-2}\,\hat{\mathbf{x}} - 2\,e^{-2}\,\hat{\mathbf{y}} = 2\,e^{-2}\left(-\hat{\mathbf{x}} - \hat{\mathbf{y}}\right)$$

which of course has the direction of (-1,-1).

b), c) Since the contours are concentric circles, the gradient is always represented by a radius line directed toward the center (where the hill is highest).

Note also that the spacing between the contours is most tightly packed approximately half way up the hill. This means that if you started at the base and walked up the hill, you would first experience a very gently sloped hill. As you climbed more the slope would become steeper, leveling out as you reached near the top of the hill.

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6. The program below shows that p (n) produces a prime number for every integer value of n between 0 and 40.

In[45]:=
```
Clear[p]
p[n_] := n^2 - n + 41
Do[If[PrimeQ[p[n]],
  Print["For n = ", n, " p[n] produces the prime number ", p[n]]], {n, 0, 40}]
```

```
For n = 0 p[n] produces the prime number 41
For n = 1 p[n] produces the prime number 41
For n = 2 p[n] produces the prime number 43
For n = 3 p[n] produces the prime number 47
For n = 4 p[n] produces the prime number 53
For n = 5 p[n] produces the prime number 61
For n = 6 p[n] produces the prime number 71
For n = 7 p[n] produces the prime number 83
For n = 8 p[n] produces the prime number 97
For n = 9 p[n] produces the prime number 113
For n = 10 p[n] produces the prime number 131
For n = 11 p[n] produces the prime number 151
For n = 12 p[n] produces the prime number 173
For n = 13 p[n] produces the prime number 197
For n = 14 p[n] produces the prime number 223
For n = 15 p[n] produces the prime number 251
For n = 16 p[n] produces the prime number 281
For n = 17 p[n] produces the prime number 313
For n = 18 p[n] produces the prime number 347
For n = 19 p[n] produces the prime number 383
For n = 20 p[n] produces the prime number 421
For n = 21 p[n] produces the prime number 461
For n = 22 p[n] produces the prime number 503
For n = 23 p[n] produces the prime number 547
For n = 24 p[n] produces the prime number 593
For n = 25 p[n] produces the prime number 641
For n = 26 p[n] produces the prime number 691
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For n = 27 p[n] produces the prime number 743
For n = 28 p[n] produces the prime number 797
For n = 29 p[n] produces the prime number 853
For n = 30 p[n] produces the prime number 911
For n = 31 p[n] produces the prime number 971
For n = 32 p[n] produces the prime number 1033
For n = 33 p[n] produces the prime number 1097
For n = 34 p[n] produces the prime number 1163
For n = 35 p[n] produces the prime number 1231
For n = 36 p[n] produces the prime number 1301
For n = 37 p[n] produces the prime number 1373
For n = 38 p[n] produces the prime number 1447
For n = 39 p[n] produces the prime number 1523
For n = 40 p[n] produces the prime number 1601
```