PHYS 301 HOMEWORK #9 - Solutions

1. a) The gradient is simply

$$\nabla \phi = \frac{\partial \phi}{\partial \mathbf{x}} \, \mathbf{\hat{x}} + \frac{\partial \phi}{\partial \mathbf{y}} \, \mathbf{\hat{y}} + \frac{\partial \phi}{\partial z} \, \mathbf{\hat{z}}$$

b & c) If you set up the relevant determinant and compute the curl by hand, you get :

$$\nabla \times (\nabla \phi) = \hat{\mathbf{x}} \left(\frac{\partial}{\partial y} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial y} \right) - \hat{\mathbf{y}} \left(\frac{\partial}{\partial x} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial}{\partial y} \frac{\partial \phi}{\partial x} \right)$$

This expression goes to zero because of the interchangeability of order of partial differentiation. In other words, the order of differentiation does not matter. Therefore, the two terms inside each parenthesis are the same, so all components are zero.

d) We write our identitity in summation notation :

The k^{th} component of the gradient can be written as:

$$abla \phi
ightarrow rac{\partial \phi}{\partial x_k}$$

And so the i^{th} component of the curl of the gradient is:

$$\nabla \times (\nabla \phi) \rightarrow \epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial \phi}{\partial x_k}$$

Now, we make use of the fact that the order of differentiation does not matter. This means we can switch the order of subscripts :

$$\epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial \phi}{\partial x_k} = \epsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial \phi}{\partial x_j}$$

But notice now that the change of subscripts causes a change of parity, which means that

$$\epsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial \phi}{\partial x_j} = -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_k}$$

This leads us to the conclusion that:

$$\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_k} = -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial \phi}{\partial x_k}$$

The only way an expression can equal its negative is if it is zero. Hence, we have a general result that $\nabla \times (\nabla \phi)$ is always zero.

2. Each cross product will produce a new vector, and we will take the dot product of these two new vectors. Let's write the left hand side of our identity in summation notation :

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) \rightarrow (\epsilon_{imn} A_m B_n)_i (\epsilon_{ijk} C_j D_k)_i$$

Remember that dot products are written in the form of $F_i G_i$, we must sum over a repeated index. This means that we need to make sure that both cross products produce the same component. Here, I have written the cross products to both produce the *i*th component of the cross product.

From here, we make use of the ϵ - δ relationship:

$$\begin{split} (\epsilon_{imn} A_m B_n)_i (\epsilon_{ijk} C_j D_k)_i \ &= \ \epsilon_{ijk} \epsilon_{imn} A_m B_n C_j D_k \ &= \ (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) A_m B_n C_j D_k \\ &= \ \delta_{jm} \delta_{kn} A_m B_n C_j D_k \ - \ \delta_{jn} \delta_{km} A_m B_n C_j D_k \end{split}$$

Now, we know that the first set of terms will be zero unless j = m and k = n, and the second cluster of terms will be zero unless j = n and k = m. We make these substitutions to get :

$$= A_m B_n C_m D_n - A_m B_n C_n D_m$$

One of the advantages of summation notation is that we are dealing with scalars, which we can multiply in any order; rearranging these terms to make the repeated indices clear we get:

$$A_{m} C_{m} D_{n} B_{n} - A_{m} D_{m} C_{n} B_{n} = (\mathbf{A} \cdot \mathbf{C}) (\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D}) (\mathbf{B} \cdot \mathbf{C})$$

3. To verify the divergence theorem, we show that :

$$\int_{\mathbf{V}} (\nabla \cdot \mathbf{v}) \, \mathrm{d}\tau = \int_{\mathbf{S}} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\mathbf{a}$$

In this problem, the volume is the cube of length two with one vertex at the origin, and the surface integral is done over the six faces of the cube. Let's do the volume integral first :

$$\nabla \cdot \mathbf{v} = \mathbf{y} + 2\mathbf{z} + 3\mathbf{x}$$

and we integrate this over the volume of the box, or:

$$\int_{V} \nabla \cdot \mathbf{v} \, d\tau = \int_{0}^{2} \int_{0}^{2} \int_{0}^{2} (y + 2z + 3x) \, dx \, dy \, dz$$

This is a very simple integral, so I will outline the steps:

$$\int_{0}^{2} \int_{0}^{2} \int_{0}^{2} (y + 2z + 3x) \, dx \, dy \, dz = \int_{0}^{2} \int_{0}^{2} \left(yx + 2zx + \frac{3}{2}x^{2} \Big|_{0}^{2} \right) dy \, dz$$
$$= \int_{0}^{2} \int_{0}^{2} (2y + 4z + 6) \, dy \, dz = \int_{0}^{2} \left(y^{2} + 4zy + 6y \right) \Big|_{0}^{2} \right) dz = \int_{0}^{2} (4 + 8z + 12) \, dz$$
$$= \left(16z + 4z^{2} \right) \Big|_{0}^{2} = 48$$

Now, to compute the surface integral, we have to compute six integral of the form :

$$\int_{S} (\mathbf{v} \cdot \mathbf{n}) \, \mathrm{da}$$

where v is the vector field, n is the unit normal to the surface, and da is the element of area on that surface. If we think a little bit about the nature of this problem, we can achieve some significant simplification.

Remember that we are computing the flux of the vector field v through each face. There will only be a flux if there is some component of v in the direction of the unit normal. For instance, to compute the flux through the face located at x = 2 (in the diagram, let's say that's the purplish side facing you). The unit normal to this face is in the + x direction; therefore, we only need to consider the x component of the dot product of (v.n). So, the flux through the plane at x = 2 is :



Note that on this face, x = 2, and the area is computed by integrating over y and z. This is a very simple integral whose value is 8

Now, what is the flux through the opposite face, i.e., the face where x = 0 always? If x = 0 everywhere on that face, the integrand is always zero and there is no flux through face at x = 0.

Two integrals down, four to go. Let's find the flux through the blue face, or the side where y = 2. The unit normal is in the + y direction, so the flux integral is :

$$\int_0^2 \int_0^2 2y \, z \, dx \, dz = 4 \int_0^2 \int_0^2 z \, dx \, dz = 16$$

Since y is zero everywhere on the opposing face, we know there is no flux through the face where y = 0.

Finally, we find the flux through the top face (z = 2) (or the beige side). The unit normal points in the plus z direction, and the flux through the top is :

$$\int_0^2 \int_0^2 3 x z \, dx \, dy = 6 \int_0^2 \int_0^2 x \, dx \, dy = 24$$

Similarly, the flux through the bottom is zero because z is zero everywhere on that face.

Adding all the individual fluxes gives us 48, verifying our volume integral.

4. We are asked to verify Stokes' theorem for the function :

$$\mathbf{f} = \mathbf{y}\,\mathbf{\hat{x}} + \mathbf{z}^2\,\mathbf{\hat{y}} + \mathbf{x}^2\,\mathbf{\hat{z}}$$

on the portion of the sphere of radius 5 lying on and above the z = -4 plane. That means we show that:

$$\int (\nabla \times \mathbf{f}) \cdot \mathbf{n} \, \mathrm{d}\mathbf{a} = \oint \mathbf{f} \cdot \mathrm{d}\mathbf{l}$$

A key point about Stokes' theorem is that you can choose any surface in the region to compute the area integral (the one on the left). The easiest way to approach the integral on the left is to find the flux of curl (that's what the integrand on the left is) through the "bottom" portion of this region, in other words, the disc lying in the z = -4 plane satisfying

$$x^2 + y^2 + z^2 = 25$$

Since we are computing the flux through a horizontal plane, we only care about the z component of the curl; the z component of the curl is simply calculated to be - 1. Therefore, the area integral for this problem becomes:

$$\int_{S} (\nabla \times \mathbf{f}) \cdot \mathbf{n} \, \mathrm{da} = -\int_{S} 1 \, \mathrm{da}$$

where we are integrating over the circle satisfying

$$x^2 + y^2 + (-4)^2 = 25$$

or

$$x^2 + y^2 = 9$$

In other words, we are integrating a constant (-1) over the circle of radius 3. This integral is trivially then (-1)(area) = -9π

Now we consider the line integral. Our obvious choice for the path is circumference of the disk lying in the z = -4 plane. We have then :

$$\int \mathbf{f} \cdot d\mathbf{l} = \int (f_x \, dx \, + f_y \, dy \, + \, f_z \, dz)$$

where we have computed the dot product of the function and dl. Since our path lies in the z = -4 plane, dz = 0 and our integral is just:

$$\int (f_x \, dx \, + f_y \, dy) \, = \, \int y \, dx \, + \, z^2 \, dy$$

Now we parameterize; the obvious parameterizations here are:

 $x = 3 \cos \theta$, $dx = -3 \sin \theta d\theta$ (remember to parameterize both x and dx)

$$y = 3\sin\theta, \, dy = 3\cos\theta \, d\theta$$

The line integral becomes:

$$\int (f_x dx + f_y dy) = \int y dx + z^2 dy = \int (3\sin\theta) (-3\sin\theta d\theta) + 16 (3\cos\theta d\theta)$$
$$= \int_0^{2\pi} (-9\sin^2\theta + 48\cos\theta) d\theta = -9\pi$$

And Stokes' Theorem survives again.

5. This problem asks you to find the flux of material through a surface. You can use the divergence theorem and compute either the volume integral :

$$\int_{\mathbf{V}} (\nabla \cdot \mathbf{v}) \, \mathrm{d}\tau$$

or the area integral:

$$\int_{S} (\mathbf{v} \cdot \mathbf{n}) \, \mathrm{da}$$

Many of you computed immediately that the divergence of the vector field is zero everywhere. This means both integrals are zero, and there is no flux through the boundary of the region. You could have done the area integral and found the same thing, but that would be a much more difficult integral to do.

6. We will use Euler's method for numerical solutions of first order differential equations to solve :

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\cos^2 y$$

with x (0) = 0 and y (0) = $\pi/4$

The crux of Euler's method is using the definition of the derivative to write :

$$f(x+h) = f(x) + hf'(x)$$

where h is the step size. In this case, we will have

$$f(x+h) = f(x) + h x \cos^2 y$$

Translating to Mathematica code :



The ODE is easily solved by separation of variables:

$$\frac{dy}{dx} = x \cos^2 y \Rightarrow \frac{dy}{\cos^2 y} = x dx$$

Integrate both sides to get :

$$\tan y = \frac{x^2}{2} + C$$

Now apply the initial conditions, when x = 0, $y = \pi/4$, so:

$$\tan \pi / 4 = 0 + C \Rightarrow C = 1$$

and our complete solution is:

$$\tan y = \frac{x^2}{2} + 1 \Rightarrow y = \arctan[(x^2 + 2)/2]$$

Now let's plot them on the same axes:



Looks good. Why does the curve of the solution approach a limit asymptotically? What is the value of this limit?

7. The date of Easter :

```
In[391]:= Clear[year, a, b, c, d, e, f, g, h, i, j, k, l, month, day, tense]
     year = RandomInteger[{1800, 2200}];
     a = Mod[year, 19];
     b = Floor[year/100];
     c = Mod[year, 100];
     d = Floor[b/4];
     e = Mod[b, 4];
     f = Floor[(b+8)/25];
     g = Floor[(b-f+1)/3];
     h = Mod[19a + b - d - g + 15, 30];
     i = Floor[c/4];
     k = Mod[c, 4];
     l = Mod[32 + 2e + 2i - h - k, 7];
     m = Floor[(a + 11h + 221)/451];
     month = Floor [(h + 1 - 7m + 114)/31];
     If[month == 3, month = " March", month = "April"];
     day = Mod[h + 1 - 7m + 114, 31] + 1;
     If[year < 2017, tense = "was", tense = "will be"];</pre>
     Print["The day of Gregorian Easter in ",
      year, " ", tense, " ", month, " ", day, ", ", year]
     The day of Gregorian Easter in 2176 will be March 31, 2176
```