PHYS 301
HOMEWORK #1

Solutions

Please read carefully the course syllabus and especially the formatting guidelines for homework. All homework assignments must be submitted in hard copy at the beginning of class on the day they are due. You must show complete work for all your solutions.

1. Consider the unit cube with opposite vertices at (0, 0, 0) and (1, 1, 1). Find the angle between the vector from the origin to (1, 1, 1) and the vector from the origin to (1, 1, 0).

Solution : There are two ways to approach this. You can use the dot product of the two vectors to find the angle. Set vector **A** = (1, 1, 1) and vector **B** = (1, 1, 0), and we have :

$$\mathbf{A} \cdot \mathbf{B} = (1, 1, 1) \cdot (1, 1, 0) = 2$$

but the dot product can also be written as:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta = 2$$

so :

$$\sqrt{3} \cdot \sqrt{2} \cos\theta = 2 \Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) = 35.2^{\circ}$$

Or just compute the tangent:

$$\tan\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 35.2^{\circ}$$

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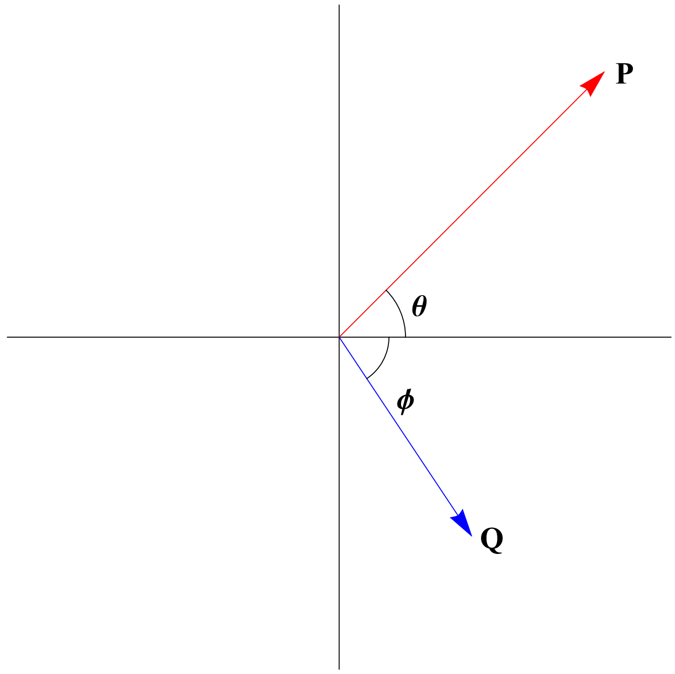
2. Position vector **P** has scalar length P and lies in the first quadrant making an angle θ above the x axis. Position vector **Q** has scalar length Q and lies in the fourth quadrant making an angle ϕ below the x axis. Use the properties of the cross product to derive the sin addition formula :

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$$

Solution : Consider the diagrram below :



We know each vector can be written :

$$\mathbf{P} = P\cos\theta\,\hat{\mathbf{x}} + P\sin\theta\,\hat{\mathbf{y}} \quad \text{and } \mathbf{Q} = Q\cos\phi\,\hat{\mathbf{x}} - Q\sin\phi\,\hat{\mathbf{y}}$$

the cross product of these two vectors can be written as:

$$|\mathbf{P}\times\mathbf{Q}| = |\mathbf{P}||\mathbf{Q}|\sin(\theta+\phi)$$

We can also compute the components of the cross product to find:

$$|\mathbf{P}\times\mathbf{Q}| = |-((P\cos\theta)(-Q\sin\phi) - (P\sin\theta)(Q\cos\phi))|$$

Equating these different formulations we get:

$$|\mathbf{P}||\mathbf{Q}|\sin(\theta+\phi) = |\mathbf{P}\,\mathbf{Q}|(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

Dividing through by the common factor PQ produces our identity.

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3. A force F acting in the x - y plane is described by :

$$\mathbf{F} = y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}$$

Find the value of the line integral for this vector over the path given by $y = x^2$ from x = 0 to x = 2.

Solution : The best way to solve this line integral is to parameterize our function and our path. If we set
x = t, we then have :

$$x = t \quad dx = dt$$

$$y = t^2 \quad dy = 2\,t\,dt$$

The line integral is:

$$\int_C \mathbf{F} \cdot d\mathbf{l}$$

where **F** is the vector valued force function, d**l** is the increment of length along the path and C is the contour of our curve, y = x^2. Writing this in Cartesian coordinates gives us:

$$\int_C (F_x\,\hat{\mathbf{x}} + F_y\,\hat{\mathbf{y}}) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}}) = \int_C (F_x\,dx + F_y\,dy) = \int_0^2 (t^2\,dt + t\,(2\,t\,dt))$$

$$= \int_0^2 3\,t^2\,dt = 8$$

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4. For all parts of this problem, surface S is the rectangle defined by y ∈ [0, 2] and z ∈ [0, 3] in the yz plane.Calculate the surface integrals

$$\int\int_S \mathbf{f} \cdot d\mathbf{A}$$

where :

a) $\mathbf{f} = 10\,\hat{\mathbf{x}}$

b) $\mathbf{f} = -10\,\hat{\mathbf{x}}$

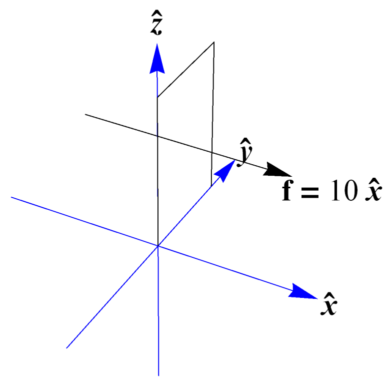
c) $\mathbf{f} = 10\,\hat{\mathbf{x}} + 6\,\hat{\mathbf{y}} - 18\,\hat{\mathbf{z}}$

d) Find a non - zero vector **f** such that the surface integral is zero. (There are both obvious and less obvious ways to accomplish this).

Five points for each part.

Solution : Before we start calculating, let' s think about what we are dealing with. Our surface is a rectangle in the yz plane, and we are calculating the flux of the vector field f through this rectangle. The element of area, dA, is a vector, whose direction is the outward normal to the surface. Since the rectangle is in the yz plane, the outward normal likes along the $\hat{\boldsymbol{x}}$ axis, and we write the element of area d**A**
as (dy dz $\hat{\boldsymbol{x}}$).

The diagram below shows how the vector **f** = 10 $\hat{\boldsymbol{x}}$ passes through the surface.



In part a), this means that there is a vector field of magnitude 10 units/area, pointing in the direction of the + x axis. The flux of this field through a rectangle of area 3×2 is then 60 units The formal way to show this is :

$$\int_0^3 \int_0^2 (10\,\hat{\mathbf{x}}) \cdot \left(\mathrm{dy}\,\mathrm{dz}\,\hat{\mathbf{x}}\right) = \int_0^3 \int_0^2 10\,\mathrm{dy}\,\mathrm{dz} = 10 \cdot 6 = 60$$

In part b), the analysis is exactly the same except the vector is pointing along the - x axis, so that the surface integral is now - 60.

The answer to part c) is the same as in part a). While the vector now has y and z components, these components lie in the plane of the rectangle, so produce no flux. Stated another way, the dot product of:

$$\left(10\,\hat{\mathbf{x}} + 6\,\hat{\mathbf{y}} - 18\,\hat{\mathbf{z}}\right) \cdot \left(\mathrm{dy}\,\mathrm{dz}\,\hat{\mathbf{x}}\right) = 60\,\mathrm{dy}\,\mathrm{dz}$$

Integrating this over the surface of the rectangle yields a value of 60 once again.

d) The obvious solution is to choose a vector that has no x component (but has y and z components). This vector then has no component perpendicular to the plane, and therefore produces no flux. Another type of solution involves a vector whose integrated x component is zero. Consider for instance, the vector :

$$\mathbf{f} = (1 - \mathrm{y})\,\hat{\mathbf{x}}$$

This vector field is in the x direction, and will be positive for $0 < y < 1$ and will be negative for $1 < y < 2$. The surface integral becomes:

$$\int_0^3 \int_0^2 (1-y)\,\hat{\mathbf{x}} \cdot \left(dy\,dz\,\hat{\mathbf{x}}\right) = \int_0^3 \int_0^2 (1-y)\,dy\,dz = \int_0^3 \left[y - \frac{y^2}{2}\right]\Big|_0^2 dz = 0$$

Similarly, a function of the form :

$$\mathbf{f} = \left(\frac{3}{2} - z\right)\hat{\mathbf{x}}$$

will also yield a zero flux.