PHYS 301
HOMEWORK #8

Due : 5 April 2017

1. Find the recursion relation and general solution near x = 2 of the differential equation :

$$y'' - (x-2)\,y' + 2\,y = 0$$

In this case, the solution will be of the form:

$$y = \sum_{n=0}^{\infty} a_n\,(x-2)^n$$

(Hint: Is there a simple substitution you can make that will simplify this problem?)

Solution : We can simplify this by setting t = x - 2. We will need to transform this equation from y (x) to y (t); this means transforming the derivatives as well. We have :

$$t = x - 2 \Rightarrow dt = dx$$

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = \frac{dy}{dt}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\,\frac{dx}{dt}\right) = \frac{d^2 y}{dt^2}$$

These steps may seem trivial (and they are), but we will need to do similar but non-trivial steps in problem 3. Our differential equation becomes:

$$y'' - t\,y' + 2\,y = 0$$

Our trial solution is:

$$y = \sum_{n=0}^{\infty} a_n\,t^n$$

Substituting the trial solution into the original equation:

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^n - t\sum_{n=1}^{\infty} n\,a_n\,t^{n-1} + 2\sum_{n=0}^{\infty} a_n\,t^n = 0$$

Multiplyint by t in the second sum, and re-indexing the first sum yields:

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2}\,t^n - \sum_{n=1}^{\infty} n\,a_n\,t^n + 2\sum_{n=0}^{\infty} a_n\,t^n = 0$$

Stripping out the n=0 term gives us:

$$2\,a_2 + 2\,a_0 = 0$$

leaving us with the recursion relation:

$$a_{n+2} = \frac{(n-2)}{(n+2)(n+1)} a_n$$

At this point, this recursion relation might seem familiar to you. It is the solution to problem 3 on HW 7. Following that solution, we have that:

$$a_2 = -a_0; \; a_4 = a_6 = a_{2n} = 0$$

and the solution was:

$$y = a_o\left(1-t^2\right) + a_1\left(t - \frac{t^3}{3!} - \frac{t^5}{5!} - \frac{3\,t^7}{7!} - \ldots\right)$$

So the solution to the original ODE in the vicinity of x = 2 is:

$$y = a_o\left(1-(x-2)^2\right) + a_1\left((x-2) - \frac{(x-2)^3}{3!} - \frac{(x-2)^5}{5!} - \ldots\right)$$

2. Consider the differential equation :

$$\left(\frac{dy}{dx}\right)^2 - y = x \quad y(0) = 1$$

Find the series solutions for this equation (there are two solutions). It is probably easier to write out the series expansion explicitly and solve for the various coefficients (rather than work with closed summation symbols). (One solution truncates quickly, the other is an infinite expansion). Find the coefficients out to a_4 for the non-truncating solution; find the entire solution for the branch that truncates quickly.

Solution : We write our trial solution :

$$y = a_o + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

Since we are told y(0) = 1, this implies that $a_o = 1$. We will use this result very soon.

$$y' = a_1 + 2\,a_2 x + 3\,a_3 x^2 + 4\,a_4 x^3 + \ldots$$

If we square dy/dx, we obtain:

$$\left(\frac{dy}{dx}\right)^2 = a_1^2 + 4\,a_2^2 x^2 + 9\,a_3^2 x^4 + 4\,a_1 a_2 x + 6\,a_1 a_3 x^2 + 8\,a_1 a_4 x^3 + 12\,a_2 a_3 x^3 + 16\,a_2 a_4 x^4 + \ldots$$

We know that we will subtract y from this expression, and equate coefficients. Let's do this term by term to save us writing down too many extraneous terms per step. We already know that a_0=1. Let's compute the coefficient of the x^0 term on the left; we have:

$$a_1^2 - a_o = 0 \Rightarrow a_1^2 = 1 \Rightarrow a_1 = \pm 1$$

The two solutions for this ODE derive from this branching. There will be one solution in which a_1=1, and another in which a_1 = -1. Now let's find the x^1coefficients:

$$4\,a_1 a_2 x - a_1 x = x \Rightarrow 4\,a_1 a_2 - a_1 = 1$$

This yields:

$$a_2 = \frac{1 + a_1}{4\, a_1} \Rightarrow a_2 = \frac{1}{4}(a_1 = 1) \text{ and } a_2 = 0\,(a_1 = -1)$$

The latter result shows us that the a_1 = -1 branch truncates quickly, and that one solution of the ODE is:

$$y_2 = a_0 + a_1 x = 1 - x$$

If you substitute this into the original ODE you will see that it satisfies the equation.

Now, let' s continue by finding expressions for a_3 and a_4. Finding the coefficients of the x^2 terms:

$$4\, a_2^2 x^2 + 6\, a_1 a_3 x^2 - a_2 x^2 = 0 \Rightarrow 4\left(\frac{1}{2}\right)^2 + 6\,(1)\, a_3 - \frac{1}{2} = 0 \Rightarrow a_3 = \frac{-1}{12}$$

Equating coefficients of x^3:

$$8\, a_1 a_4 x^3 + 12\, a_2 a_3 x^3 - a_3 = 0 \Rightarrow 8\,(1)\, a_4 + 12\left(\frac{1}{2}\right)\left(\frac{-1}{12}\right) + \frac{1}{12} = 0 \Rightarrow \frac{5}{96}$$

And the second solution to this ODE is:

$$y = 1 + x + \frac{x^2}{2} - \frac{x^3}{12} + \frac{5\,x^4}{96} + \ldots$$

Below is a short *Mathematica* program that will compute as many coefficients as we wish:

```
Clear[f, c, a, x]
f[x_] := Sum[a[m] x^m, {m, 0, 14}]
(*We make use of the fact that we know a0=
 1 and a1 = 1 for the infinite series *)
a[0] = 1; a[1] = 1;
c = CoefficientList[Collect[D[f[x], x]^2 - f[x], x], x];
Solve[{c[[2]] == 1, c[[3]] == 0, c[[4]] == 0, c[[5]] == 0, c[[6]] == 0,
  c[[7]] == 0, c[[8]] == 0, c[[9]] == 0, c[[10]] == 0, c[[11]] == 0, c[[12]] == 0},
 {a[2], a[3], a[4], a[5], a[6], a[7], a[8], a[9], a[10], a[11], a[12]}]
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Out[277]= $\left\{\left\{a[2] \to \frac{1}{2},\ a[3] \to -\frac{1}{12},\ a[4] \to \frac{5}{96},\ a[5] \to -\frac{41}{960},\ a[6] \to \frac{469}{11\,520},\ a[7] \to -\frac{6889}{161\,280},\ a[8] \to \frac{24\,721}{516\,096},\ a[9] \to -\frac{2\,620\,169}{46\,448\,640},\ a[10] \to \frac{64\,074\,901}{928\,972\,800},\ a[11] \to -\frac{1\,775\,623\,081}{20\,437\,401\,600},\ a[12] \to \frac{1\,571\,135\,527}{14\,014\,218\,240}\right\}\right\}$

3. The Legendre differential equation is :

$$\left(1 - x^2\right) y'' - 2\, x\, y' + m\,(m+1)\, y = 0$$

Set x = cos θ and show that the equation becomes :

$$\frac{d^2 y}{d\theta^2} + \cot\theta \frac{dy}{d\theta} + m(m+1)\,y = 0$$

Solution : The key here is to transform the equation from y (x) to y (θ). This will require converting every term involving x, including the derivates, to terms involving y (θ). The coefficient of y" becomes :

$$\left(1 - x^2\right) = \left(1 - \cos^2\theta\right) = \sin^2\theta$$

We use the chain rule to convert dy/dx into dy/dθ:

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}$$

since $x = \cos\theta$, we know that $dx = \frac{-1}{\sin\theta} d\theta$, so that we can write :

$$\frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{-1}{\sin\theta}\right) \equiv w$$

Now, we find an expression for tranforming y" (x) to y" (θ). Start with :

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dw}{dx}$$

where we have defined w above. Using the chain rule again:

$$\frac{dw}{dx} = \frac{dw}{d\theta}\frac{d\theta}{dx}$$

Using the definition of w:

$$\frac{dw}{d\theta} = \frac{d}{d\theta}\left(\frac{dy}{d\theta} \cdot \frac{-1}{\sin\theta}\right) = \frac{-1}{\sin\theta}\frac{d^2 y}{d\theta^2} + \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta}$$

Remember, this is just dw/dθ, so the total second derivative becomes :

$$\frac{dw}{dx} = \frac{dw}{d\theta}\frac{dx}{d\theta} = \left(\frac{-1}{\sin\theta}\frac{d^2 y}{d\,\theta^2} + \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\,\theta}\right)\left(\frac{-1}{\sin\theta}\right)$$

Now, putting all these terms together; the second derivative term becomes :

$$\left(1 - x^2\right) y" \to \sin^2\theta\left(\frac{-1}{\sin\theta}\frac{d^2 y}{d\theta^2} + \frac{\cos\theta}{\sin^2\theta}\frac{dy}{d\theta}\right)\left(\frac{-1}{\sin\theta}\right)$$

$$= \frac{d^2 y}{d\,\theta^2} - \cot\theta\frac{dy}{d\theta}$$

The first derivative term becomes:

$$-2\,x\,y' \to -2\cos\theta\left(\frac{-1}{\sin\theta}\right)\frac{dy}{d\theta} = +2\cot\theta\frac{dy}{d\theta}$$

Since the final term does not involve x explicitly, it is not changed, combining all terms we get:

$$\left(1-x^2\right) y'' - 2\,x\,y' + m\,(m+1)\,y \rightarrow \frac{d^2 y}{d\theta^2} + \cot\theta \frac{dy}{d\theta} + m\,(m+1)\,y = 0$$

4. Problem 12.64 from p. 679 of the text. All parts (part a = 10 points, part b = 20 pts, c and d = 10 pts each).

a) $y'' - 2\,x\,y' + 2\,k\,y = 0$

Using a trial solution of :

$$y = \sum_{n=0}^{\infty} a_n x^n$$

our differential equation becomes:

$$\sum_{n=2}^{\infty} n\,(n-1)\,a_n\,x^{n-2} - 2\,x \sum_{n-1}^{\infty} n\,a_n\,x^{n-1} + 2\,k \sum_{n=0}^{\infty} a_n\,x^n = 0$$

Re - indexing the first sum and multiplying by x in the second :

$$\sum_{n=0}^{\infty} (n+1)\,(n+2)\,a_{n+2}\,x^n - 2 \sum_{n-1}^{\infty} n\,a_n\,x^n + 2\,k \sum_{n=0}^{\infty} a_n\,x^n = 0$$

Stripping out the n = 0 terms:

$$2\,a_2 + 2\,k\,a_o = 0 \Rightarrow a_2 = -\,k\,a_o$$

and the recursion relation becomes:

$$a_{n+2} = \frac{2\,(n-k)\,a_n}{(n+2)\,(n+1)}$$

b) In the specifc case where $a_o = 0$, k=5 and a_1=15:

i) If $a_o = 0$ then a_2 and all other even coefficients are 0.

ii) $a_3 = \dfrac{2\,(1-5)}{3 \cdot 2} a_1 = \dfrac{-4}{3} \cdot 15 = -20$

$$a_5 = 2\,\frac{(3-5)\,(-20)}{5 \cdot 4} = 4$$

iii) The n - k term will cause the n = 5 term to go to zero, and since all odd terms are related, all higher order odd coefficients must also be zero.

iv) $y = 15\,x - 20\,x^3 + 4\,x^5$

Substitute into the original ODE :

$$80\,x^3 - 120\,x - 2\,x\left(20\,x^4 - 60\,x^2 + 15\right) + 2\,(5)\left(4\,x^5 - 20\,x^3 + 15\,x\right)$$

$$= 40\,x^5 - 40\,x^5 + 80\,x^3 + 120\,x^3 - 200\,x^3 - 120\,x - 30\,x + 150\,x = 0$$

c) Even/odd values of k produce truncated even/odd polynomials, since the n - k term will be zero, and therefore all higher order even/odd terms will be zero.

d) We need k = 4 and a_1 =0, then we have:

$$a_2 = \frac{2\,(0-4)\,a_o}{2\cdot 1} = -48$$

$$a_4 = \frac{2\,(2-4)\,(-48)}{4\cdot 3} = 16$$

$$y = 12 - 48\,x^2 + 16\,x^4$$