PHYS 314
SECOND HOUR EXAM
SOLUTIONS

1. The most common error I have seen so far is that students are treating potential as a vector and taking only the y component. Potential is a scalar and has no components. The potential at the point b from the midpoint of the rod is :

$$d\Phi = -\frac{G\,dm}{r}$$

$$dm = \rho\,dx \quad \text{and } r = \sqrt{x^2 + b^2}$$

Therefore, integrating along the entire rod gives :

$$\Phi = -2G\rho\int_0^L \frac{dx}{\sqrt{x^2+b^2}} = -2G\rho\left[\ln\left(x^2+b^2\right)+x\right]\Big|_0^L = -2G\rho\left[\ln\left(L^2+b^2\right)+L-\ln(b)\right]$$

The limits of integration and factor of 2 make use of the symmetry of the situation. We find the force from:

$$dF = -\frac{G\,m\,dm}{r^2}$$

where r and dm will be as above. However, since forces are vectors, the symmetry of the situation tells us that only the y component of force will be non-zero. Defining θ as the angle between the vertical and a line from b to the rod, we have that the y component of the force is written as:

$$dF_y = -\frac{G\,m\,dm\cos\theta}{r^2} = -\frac{G\,m\,\rho\,dx}{x^2+b^2}\left(\frac{b}{\sqrt{x^2+b^2}}\right)$$

$$\text{and } F_y = -2G\,m\,\rho\,b\int_0^L \frac{dx}{\left(x^2+b^2\right)^{3/2}} = -2G\,m\,\rho\,b\left(\frac{x}{b^2\sqrt{x^2+b^2}}\right)\Bigg|_0^L$$

2. a) This is done in detail in both Felder and Felder and Thornton/Marion, although I think the treatment in Felder and Felder is much clearer.

b) The length of a path on a surface is simply :

$$J = \int_a^b ds$$

On a cylinder, the element of length is found from :

$$ds^2 = d\rho^2 + \rho^2 \, d\phi^2 + dz^2$$

Since ρ is constant on a cylinder, $d\rho = 0$, and we have :

$$ds = \sqrt{\rho^2 \, d\phi^2 + dz^2} = dz \sqrt{1 + \rho^2 \, (d\phi / dz)^2}$$

so that

$$J = \int_a^b \sqrt{1 + \rho^2 \, (d\phi / dz)^2} \; dz$$

and we need to find the function that minimizes the integral above. Therefore, we use the integrand in the Euler-Lagrange equation:

$$\frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right) - \frac{\partial f}{\partial \phi} = 0$$

where

$$f = \sqrt{1 + \rho^2 \, (d\phi / dz)^2} \text{ and } \phi' = d\phi / dz$$

Since there is no explicit ϕ dependence of f, we know that

$$\frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right) = 0 \Rightarrow \frac{\partial f}{\partial \phi'} = cst$$

or :

$$\frac{\partial}{\partial \phi'} \sqrt{1 + \rho^2 \, (\phi')^2} = \frac{\rho^2 \, \phi'}{\sqrt{1 + \rho^2 \, (\phi')^2}} = C$$

Remembering that ρ is a constant on the surface of a cylinder, simple algrebra will yield:

$$\frac{d\phi}{dz} = c \Rightarrow \phi = a\,z + b$$

which is the equation for a helix. If you had written the solution in terms of $z(\phi)$, you would have followed exactly the same procedure and obtained $z = c\,\phi + d$

3. Let' s start by writing x (t) and y (t); if θ is the angle between the vertical and the pendulum, we have :

$$x(t) = b \sin\theta \quad \Rightarrow \dot{x} = b \, \dot{\theta} \cos\theta$$

$$y(t) = \frac{1}{2} a\, t^2 - b \cos\theta \Rightarrow \dot{y} = a\,t + b\,\dot{\theta} \sin\theta$$

$$\text{then, } T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\left(a^2\, t^2 + b^2\, \dot{\theta}^2 + 2\,a\,b\,t\,\dot{\theta} \sin\theta\right)$$

$$U = m g y = m g\left(\frac{1}{2} a t^2 - b\cos\theta\right)$$

Then, we can write the Lagrangian :

$$L = T - U = \frac{m}{2}\left(a^2 t^2 + b^2 \dot{\theta}^2 + 2 a b t \dot{\theta} \sin\theta\right) - m g\left(\frac{1}{2} a t^2 - b\cos\theta\right)$$

and we see we can write the entire Lagrangian in terms of θ. Taking partial derivatives :

$$\frac{\partial L}{\partial \dot{\theta}} = m b^2 \dot{\theta} + m a b t \sin\theta \quad \frac{\partial L}{\partial \theta} = m a b t \dot{\theta} \cos\theta - m g b \sin\theta$$

and since

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

Taking the time derivative :

$$\frac{d}{dt}\left(m b^2 \dot{\theta} + m a b t \sin\theta\right) = m b^2 \ddot{\theta} + m a b \sin\theta + m a b t \dot{\theta} \cos\theta$$

Equating terms:

$$m b^2 \ddot{\theta} + m a b \sin\theta + m a b t \dot{\theta} \cos\theta = m a b t \dot{\theta} \cos\theta - m g b \sin\theta$$

Which yields:

$$m b^2 \ddot{\theta} + m b (a + g) \sin\theta = 0$$

and finally

$$\ddot{\theta} + \frac{(a + g)}{b} \sin\theta = 0$$

In the small angle approximation, $\sin\theta \approx \theta$ so

$$\ddot{\theta} + \frac{(a + g)}{b} \theta = 0$$

But this is the standard harmonic oscillator equation, where the period is given by

$$P = \frac{2\pi}{\omega}$$

$$\text{where } \omega^2 = \frac{a + g}{b}$$

Thus, the period of oscillation is

$$P = 2\pi \sqrt{\frac{b}{a + g}}$$

which reduces to $2\pi \sqrt{b/g}$ if the acceleration is zero, identical with the standard equation of the period of a pendulum (in the small angle approximation).