

# PHYS 314

## HOMework #1

### Solutions

Now that you've completed more advanced physics and math, let's refresh and review some basic of Newtonian mechanics. Make sure you have read the syllabus and the guidelines for submitting homework.

1. A projectile is launched from level ground. At what angle should it be launched so that its maximum elevation equals its range. (Neglect air friction).

**Solution** : Let's start with the equations of motion neglecting air friction :

$$x(t) = v_0 \cos \theta t$$
$$y(t) = v_0 \sin \theta t - \frac{1}{2} g t^2$$

There are a number of ways of determining the maximum height of the projectile. One of these many ways is to set the first derivative of  $y(t) = 0$ :

$$\frac{dy}{dt} = v_0 \sin \theta - g t = 0 \Rightarrow t = \frac{v_0 \sin \theta}{g}$$

This tells us the time of maximum height; substitute this into  $y(t)$  to find:

$$y_{\max} = \frac{v_0^2 \sin^2 \theta}{2g}$$

In the absence of air friction, we expect the total time of flight to be twice the time to maximum height; using this value in  $x(t)$  gives:

$$\text{range} = v_0 \cos \theta \left( \frac{2 v_0 \sin \theta}{g} \right) = \frac{2 v_0^2 \sin \theta \cos \theta}{g}$$

If we equate the expressions for maximum height and range we get:

$$\frac{v_0^2 \sin^2 \theta}{2g} = \frac{2 v_0^2 \sin \theta \cos \theta}{g} \Rightarrow \tan \theta = 4 \Rightarrow \theta \approx 76^\circ$$

---

2. A projectile is fired on level ground with initial velocity  $v$  and follows a parabolic trajectory (i.e., neglect air friction). It passes through two points that are both a distance  $h$  above the horizon. Show that if the gun is adjusted for maximum range, the distance between the two points is :

$$d = \frac{v_0}{g} \sqrt{v_0^2 - 4gh}$$

**Solution** : From our range equation in question 1, it should be clear that the maximum range (in the absence of friction) occurs when  $\theta = 45^\circ$

In this particular problem, we are more concerned with values of  $y(x)$  rather than  $y(t)$  or  $x(t)$ , so remembering back to first semester physics, we write  $x(t)$  in terms of  $t$ :

$$t = \frac{x}{v_0 \cos \theta}$$

and use this value of  $t$  in the  $y(t)$  equation to get our  $y(x)$  relationship:

$$y(x) = x \tan \theta - \frac{g x^2}{2 v_0^2 \cos^2 \theta}$$

Now, if  $\theta = 45^\circ$ ,  $\tan \theta = 1$  and  $2 \cos^2 \theta = 1$ , so our  $y(x)$  for this situation simplifies to:

$$y(x) = x - \frac{g x^2}{v_0^2}$$

We know that  $y(x) = h$  at two points, and we can solve the quadratic equation:

$$h = x - \frac{g x^2}{v_0^2}$$

to find the two values of  $x$  at which  $y(x) = h$ . Solving this equation gives us:

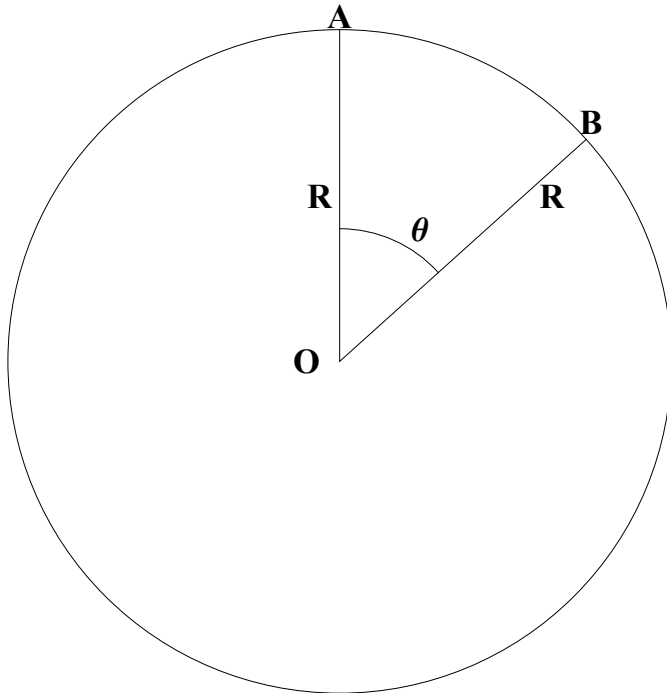
$$x = \frac{1 \pm \sqrt{1 - 4 \frac{gh}{v_0^2}}}{2 \frac{g}{v_0^2}}$$

We are asked to find the distance between these two points, so we merely need to subtract one solution from the other, which produces:

$$x_2 - x_1 = \frac{2 \sqrt{1 - 4 \frac{gh}{v_0^2}}}{2 \frac{g}{v_0^2}} = \frac{v_0^2 \sqrt{\frac{v_0^2 - 4gh}{v_0^2}}}{g} = \frac{v_0}{g} \sqrt{v_0^2 - 4gh}$$

3. A bead of mass  $m$  is initially stationary at the top of a sphere of radius  $R$ . The bead begins to slide down the frictionless surface of the sphere. Determine the angle  $\theta$  with respect to the vertical that the bead leaves the sphere.

**Solution** : This is a problem that combines concepts of energy conservation, Newton's second law, and circular motion.



Let's consider the particle at points A and B. At A, we know there is no kinetic energy; if we use the center of the circle as our reference level, there is  $m g R$  potential energy at A. At point B, some of the potential has been converted to kinetic energy, and we can write the energy balance as :

$$m g R = \frac{1}{2} m v_B^2 + m g R \cos \theta$$

where  $v_B$  is the speed at B, and  $m g R \cos \theta$  is the PE at B. This allows us to write the speed of the particle at B as:

$$v_B^2 = 2 g R (1 - \cos \theta)$$

The particle will leave the sphere when the normal force of the sphere on the particle is zero. To find an expression for the normal force, we consider forces along the radial line from O to B. The normal force acts radially outward at B, and the component of gravity along the radius is  $m g \cos \theta$ . These forces combine to produce a centripetal force acting inward along the radius; we express Newton's second law as :

$$N - m g \cos \theta = - \frac{m v_B^2}{R} \Rightarrow N = m \left( g \cos \theta - \frac{v_B^2}{R} \right)$$

The term on the right is negative since the centripetal force points inward (which we have chosen as the negative direction.) The particle will leave the sphere when  $N = 0$ , or when

$$g \cos \theta = \frac{v_B^2}{R}$$

According to our energy balance::

$$v_B^2 = 2 g R (1 - \cos \theta)$$

so the particle leaves the sphere when:

$$g \cos \theta = \frac{2 g R (1 - \cos \theta)}{R} \Rightarrow \cos \theta = 2/3 \Rightarrow \theta = 48.2^\circ$$

4. A boat can travel with a speed of  $v$  in still water. If the boat is now in a river flowing with constant current speed of  $V$ , show that the time to go a distance  $D$  upstream and return to the starting point is :

$$t = \frac{2 D v}{v^2 - V^2}$$

what is the significance of the minus sign if  $V > v$ ? If  $V = v$ ?

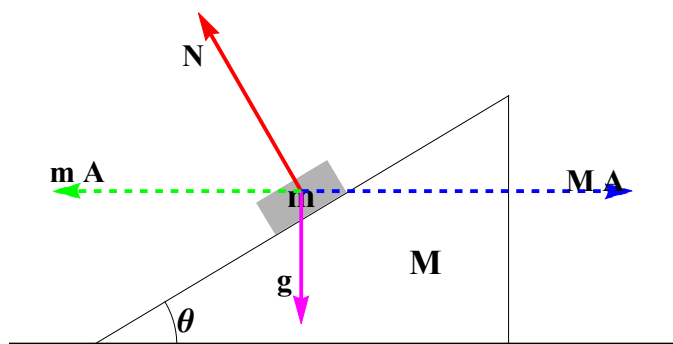
**Solution :** We find the time to make each leg of the trip, and sum them to find the total time. Going upstream (against the current), the speed of the boat is  $v - V$ , so the time to go a distance upstream is  $D/(v - V)$ . Going downstream, the speed with the current is  $v + V$ , so the time to make the return trip is simply  $D/(v + V)$ . The total time is then :

$$\frac{D}{v - V} + \frac{D}{v + V} = \frac{D(v + V) + D(v - V)}{(v + V)(v - V)} = \frac{2 D v}{(v + V)(v - V)} = \frac{2 D v}{v^2 - V^2}$$

If  $V = v$ , the boat never makes the return trip. It will take an infinite amount of time for the boat to get to the starting position, and the corresponding time of travel is infinite. If  $V > v$ , the boat would be swept backwards.

5. (This one's a little challenging). The inclined side of a wedge of mass  $M$  makes an angle  $\theta$  with the horizontal. A block of mass  $m$  slides down this incline. All surfaces are frictionless (meaning the wedge slides without friction on the surface). Find the acceleration of the small mass  $m$ , and the acceleration of the wedge of mass  $M$ . Solve this using Newton's laws. We will investigate this problem later using Lagrangian dynamics.

Solution : Let's start by considering the diagram below :



The diagram shows the force of gravity acting on the smaller block  $m$ , the normal force between the wedge  $M$  and block  $m$ , and the direction of acceleration of  $M$ . The green dashed line represents

the "fictitious force" acting on  $m$ , since  $m$  is sliding down a wedge accelerating to the right.

We want to use Newton's laws to write out enough equations of motion that we can solve for the accelerations of the block and the wedge. (We will need three such equations).

We can get the first of by considering forces on  $M$ . We know that  $m$  exerts a normal force of magnitude  $N$  on the wedge  $M$ . The horizontal component of this normal force will be the force causing  $M$  to accelerate. The component of  $N$  acting in the direction of  $A$  is  $N \sin \theta$ , so we can write Newton's second Law for  $M$  as :

$$N \sin \theta = M A \quad (1)$$

Unfortunately, this equations involves two unknowns, so we know we will need at least another two equations (one to solve for  $N$  and one to solve for  $a$ , the acceleration of the block  $m$ ).

Let's consider the forces acting down the plane on block  $m$ . We know the component of gravity acting down the plane is  $m g \sin \theta$ . The component of the fictitious force acting down the plane is  $m A \cos \theta$ . These forces combine to produce an acceleration down the plane given by:

$$m a = m A \cos \theta + m g \sin \theta \Rightarrow a = A \cos \theta + g \sin \theta \quad (2)$$

For our final equation, we recognize that the block is not accelerating in the direction perpendicular to the plane. This means that the sum of all forces perpendicular to the incline must be zero. Applying Newton's second law we obtain:

$$N - m g \cos \theta + m A \sin \theta = 0$$

Equation (3) allows us to express the normal force :

$$N = m g \cos \theta - m A \sin \theta \quad (3)$$

Substituting this into Equation (1):

$$(m g \cos \theta - m A \sin \theta) \sin \theta = M A \Rightarrow m g \sin \theta \cos \theta = A (M + m \sin^2 \theta)$$

or

$$A = \frac{m g \sin \theta \cos \theta}{M + m \sin^2 \theta} \quad (4)$$

Equation (4) is the acceleration of the wedge. We can get the acceleration of the block by substituting this value for  $A$  into eq. (2) :

$$\begin{aligned} a &= \frac{m g \sin \theta \cos \theta}{M + m \sin^2 \theta} \cos \theta + g \sin \theta = g \sin \theta \left[ \frac{m \cos^2 \theta}{M + m \sin^2 \theta} + 1 \right] \\ &= g \sin \theta \left[ \frac{m \cos^2 \theta + (M + m \sin^2 \theta)}{M + m \sin^2 \theta} \right] = g \sin \theta \left[ \frac{m + M}{M + m \sin^2 \theta} \right] \end{aligned}$$

Notice that if  $M$  grows very large (and therefore very difficult to accelerate), the expression in the brackets approaches 1, and the acceleration of the block approaches its familiar value of  $g \sin \theta$ .