PHYS 314
HOMEWORK #6-- Solutions

1. We are asked to show that the energy lost over one cycle of a damped oscillator is given by :

$$\int_0^{2\pi/\omega} (2\,m\,\beta\,\dot{x}) \cdot (\dot{x}\,dt) = 2\,\pi\,m\,\omega\,\beta\,D^2$$

The solution for the driven oscillator is given as:

$$x(t) = D\cos(\omega\,t - \delta) \tag{1}$$

where x, ω, t, and δ have their common meanings in oscillatory problems. The total energy of an oscillator is given by:

$$E = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,k\,x^2 \tag{2}$$

We also know that the equation of motion for a damped oscillator is given by:

$$m\,\ddot{x} + b\,\dot{x} + k\,x = 0 \tag{3}$$

where b is defined as 2 m β.

We begin by taking the time derivative of eq. (2):

$$\frac{dE}{dt} = m\,\dot{x}\,\ddot{x} + k\,x\,\dot{x} = \dot{x}\,(m\,\ddot{x} + k\,x) \tag{4}$$

Note that we can rewrite eq. (3) as:

$$2\,m\,\beta\,\dot{x} = -(m\,\ddot{x} + k\,\dot{x})$$

so that we can rewrite eq. (4) as:

$$\frac{dE}{dt} = -\dot{x}\,(2\,m\,\beta\,\dot{x}) \Rightarrow E = \int_0^{2\pi/\omega} (2\,m\,\beta\,\dot{x}) \cdot (\dot{x}\,dt)$$

Since x (t) = D cos (ω t - δ), we have that :

$$\dot{x}(t) = -\omega\,D\sin(\omega\,t - \delta)$$

and the integral becomes :

$$E = 2\,m\,\beta\,\omega^2\,D^2 \int_0^{2\pi/\omega} \sin^2(\omega\,t - \delta)\,dt = 2\,\pi\,m\,\omega\,\beta\,D^2$$

since the integral evaluates to π/ω:

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Integrate[Sin[ω t - δ]^2, {t, 0, 2 π / ω}]
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$$\frac{\pi}{\omega}$$

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2. We write the following code :

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Clear[x, A, β, t, ω1, δ, v, ω0]
A = 1; ω0  =  1; β = 0.1; δ = π;
ω1  =  Sqrt[ω0^2 - β^2];

x[t_] := A Exp[-β t] Cos[ω1 t - δ]
v[t_] := -A Exp[-β t] (β Cos[ω1 t - δ] + ω1 Sin[ω1 t - δ])
Plot[{x[t], v[t]}, {t, 0, 50}, PlotRange → All, PlotLegends → {"x[t]", "v[t]"}]
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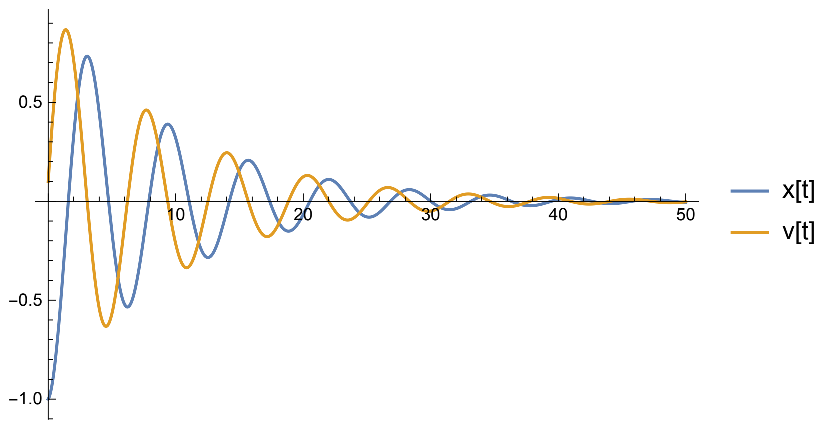

and for the phase diagram :

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ParametricPlot[{x[t], v[t]}, {t, 0, 50}, PlotRange → All]
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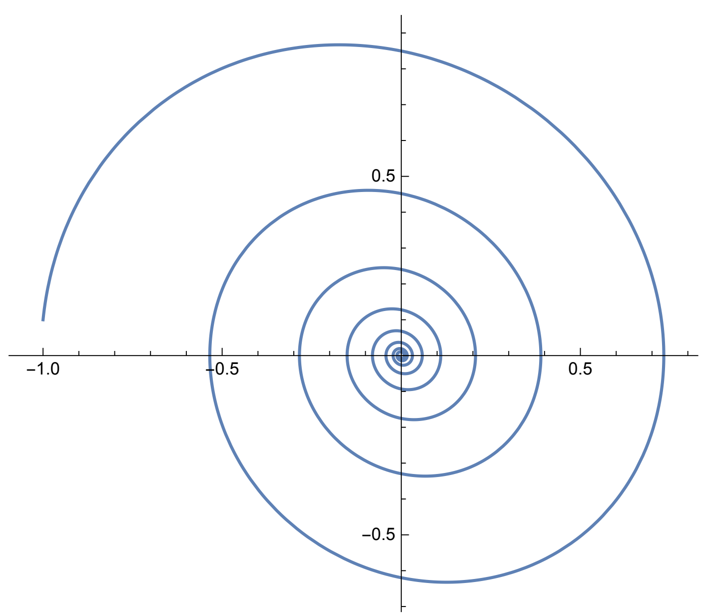

3. Find the Taylor expansion of :

$$\frac{1}{\sqrt{1-k^2\,x^2}}$$

To make our lives simple, you will follow the hint given in class Monday and set $z = k^2\,x^2$ so that you can vastly simplify your differentiations. We know that the Taylor expansion will be:

$$f\,(z) \;=\; f\,(0) + \frac{f'\,(0)\,z}{1\,!} + \frac{f''\,(0)\,z^2}{2\,!} + \frac{f'''\,(0)\,z^3}{3\,!} + \ldots$$

Evaluating these terms :

$$f\,(z) \;=\; \frac{1}{\sqrt{1-z}} \;\Rightarrow\; f\,(0) \;=\; 1$$

$$f'\,(z) \;=\; \frac{-1}{2}\,(1-z)^{-3/2}\,(-1) \;=\; \frac{1}{2}\,(1-z)^{-3/2} \Rightarrow\; f'\,(0) \;=\; \frac{1}{2}$$

$$f''\,(z) \;=\; \frac{-3}{2}\cdot\frac{1}{2}\,(1-z)^{-5/2}\,(-1) \;=\; \frac{3}{4}\,(1-z)^{-5/2} \;\Rightarrow\; f''\,(0) \;=\; \frac{3}{4}$$

$$f'''\,(z) \;=\; \frac{15}{8}\,(1-z)^{-7/2} \Rightarrow\; f'''\,(0) \;=\; \frac{15}{8}$$

$$f^{iv}\,(z) \;=\; \frac{105}{16}\,(1-z)^{-9/2} \;\Rightarrow\; f^{iv}\,(0) \;=\; \frac{105}{16}$$

Using these values in the expansion:

$$f\,(z) \;=\; 1 + \frac{1}{2}\,z + \frac{3}{4}\,\frac{z^2}{2\,!} + \frac{15}{8}\,\frac{z^3}{3\,!} + \frac{105}{16}\,\frac{z^4}{4\,!} + \ldots$$

$$= \;1 + \frac{1}{2}\,z + \frac{3}{8}\,z^2 + \frac{5}{16}\,z^3 \;+\; \frac{35}{128}\,z^4 + \ldots$$

Now, set $z \;=\; k^2\,x^2$:

$$f\,(x) \;=\; 1 + \frac{k^2\,x^2}{2} + \frac{3}{8}\,k^4\,x^4 \;+\; \frac{5}{16}\,k^6\,x^6 \;+\; \frac{35}{128}\,k^8\,x^8 + \ldots$$

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4. For this problem, we envision a protostar of radius r and mass m which accretes matter until it grows to its final configuration of mass M and radius R. We model the accretion by assuming thin spherical shells of radius r and thickness dr are added to the edge of the underlying protostar. The gravitational potential between two masses is :

$$\Phi \;=\; -\,\frac{G\,m_1\,m_2}{r}$$

Here, we will call the underlying protostar the first object, and the spherical shell the second object. The mass of the star is :

$$m_* = \frac{4}{3} \pi r^3 \rho$$

and the mass of the shell is:

$$m_{sh} = 4 \pi r^2 \pi \, dr$$

since the volume of the shell is surface area x thickness.

Using these expressions for mass, we get that:

$$d\Phi = -\frac{G\left(\frac{4}{3} \pi r^3 \rho\right)\left(4 \pi r^2 \rho \, dr\right)}{r} = \frac{-16}{3} G \pi^2 \rho^2 r^4 \, dr$$

The total gravitational potential is obtained by integrating dΦ from 0 to R:

$$\Phi = \frac{-16}{15} G \pi^2 \rho^2 R^5$$

the density can be written as:

$$\rho = \frac{M}{V} = \frac{3 M}{4 \pi R^3} \Rightarrow \rho^2 = \frac{9 M^2}{16 \pi^2 R^6}$$

Using this value in the Φ equation:

$$\Phi = \frac{-16}{15} G \left(\frac{9 M^2}{16 \pi^2 R^6}\right) \pi^2 R^5 = \frac{-3}{5} \frac{G M^2}{R}$$

For the Earth, $M_\oplus = 6\ 10^{24}$kg, $R_\oplus = 6.4\ 10^6$m

so that the total gravitational binding energy of the Earth in SI units (Joules) is:

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Clear[G, M, R, Φ]
M = 6 × 10^24; R = 6.4 × 10^6; G = 6.67 × 10^-(11);
Φ = -0.6 G M^2 / R
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-2.25113×10^{32}

Thirty seconds on the internet will reveal that a one megaton nuclear weapon releases 4.2 petajoules of energy, so that the total destructive power of 60,000 such weapons is:

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6 × 10^4 (4. × 10^15)
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2.4×10^{20}

not even close.

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5. For this problem, we imagine that the larger sphere does not have a hollowed out region, but that it consists entirely of material of density ρ. We then imagine the smaller sphere contains material of density - ρ, so that the positive and negative matter cancel out, leaving no net mass in the smaller sphere.

Then, since the point lies external to the sphere, we know the potential due to each "mass" acts as if all its mass is located at a point in the center of the sphere.

Thus, the gravitational potential due to the larger sphere is :

$$\Phi_1 = -G \frac{\left(\frac{4}{3}\pi\rho\ a^3\right)}{X}$$

and the potential due to the hollowed sphere is:

$$\Phi_2 = \frac{-G\left(\frac{4}{3}\pi(-\rho)\,b^3\right)}{X-\frac{a}{2}}$$

and the total potential at X is the sum of these two potentials.