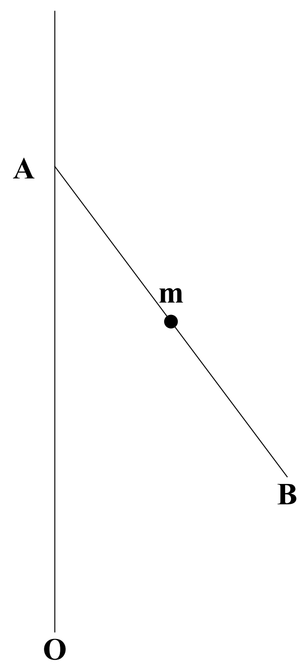
PHYS 314
HOMEWORK #9

Due : 12 April 2017

1. AB is a straight frictionless wire fixed at point A on a vertical axis OA such that AB rotates around OA at a constant angular velocity ω. A bead of mass m is constrained to move along this wire. Set up the Lagrangian, write Lagrange' s equations, and determine the motion of m at any time.



Solution : We begin by writing expressions for the kinetic and potential energy of the mass. The mass can move radially along the wire, and also has rotational motion around OA. Since the wire is fixed at a constant angle α, we can write the kinetic energy of the particle as :

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2 \alpha \, \omega^2\right)$$

Let' s make sure we understand this expression for kinetic energy. The first term is the energy due to the radial motion of the particle along the wire. The second term is the energy due to its rotational motion. We know that the linear speed of a rotating object is R ω. Here, remember that the instantaneous radius of orbit is r sin α. Since the angular velocity is constant, we do not need to use

θ explicitly as a generalized coordinate, so that we can express the Lagrangian entirely in one degree of freedom, r. If we use A as the reference level U = - m g r cos α. The Lagrangian is :

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2 \alpha\, \omega^2\right) - (-\, m\, g\, r \cos \alpha)$$

The Lagrangian equations become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}(m\, \dot{r}) - \left(m\, r \sin^2 \alpha\ \omega^2 + m\, g \cos \alpha\right) = 0$$

Which leads to the equation of motion:

$$\ddot{r} - r \sin^2 \alpha\, \omega^2 - g \cos \alpha = 0$$

By rewriting this as:

$$\ddot{r} - \left(\omega^2 \sin^2 \alpha\right) r = g \cos \alpha$$

we recognize this to be a very simple differential equation. The homogeneous solution will be simply

$$r = c_1\, e^{\omega \sin \alpha\, t} + c_2\, e^{-(\omega \sin \alpha)\, t} = c_3 \cosh (\omega \sin \alpha\, t) + c_4 \sinh (\omega \sin \alpha\, t)$$

The particular solution is $r_p = -\, g \cos \alpha/(\omega \sin \alpha)^2$

so the complete solution is :

$$r(t) = c_3 \cosh (\omega \sin \alpha\, t) + c_4 \sinh (\omega \sin \alpha\, t) - \frac{g \cos \alpha}{(\omega \sin \alpha)^2}$$

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2. A bead slides without friction on a wire in the shape of a cycloid, which can be parameterized as :

$$x = a\,(\theta - \sin\theta) \quad y = a\,(1 + \cos \theta)$$

where a is a constant. Find the Lagrangian for the system and write Lagrange’s equations.

Solution : Hopefully these steps are becoming second nature :

$$\dot{x} = a\left(\dot{\theta} - \dot{\theta} \cos \theta\right) \qquad \dot{y} = a\, \dot{\theta} \sin \theta$$

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2} m\left[a^2 \left(\dot{\theta}^2 - 2\, \dot{\theta}^2 \cos \theta + \dot{\theta}^2 \cos^2 \theta\right) + a^2\, \dot{\theta}^2 \sin^2 \theta\right] = \frac{1}{2} m\, a^2\left[2\, \dot{\theta}^2\, (1 - \cos \theta)\right]$$

$$U = m\, g\, y = m\, g\, a\,(1 + \cos \theta)$$

By writing T and U in terms of θ, we need only one generalized coordinate. Therefore, the Lagrangian is :

$$L = \frac{1}{2} m\, a^2\left[2\, \dot{\theta}^2\, (1 - \cos \theta)\right] - m\, g\, a\,(1 + \cos \theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = 2\, m\, a^2\, \dot{\theta}\,(1 - \cos \theta) \quad \frac{\partial L}{\partial \theta} = m\, a^2\, \dot{\theta}^2 \sin \theta + m\, g\, a \sin \theta$$

and we obtain:

$$\frac{d}{dt}\left(2\, m\, a^2\, \dot{\theta}\, (1 - \cos\theta) \right) - \left(m\, a^2\, \dot{\theta}^2 \sin\theta + m\, g\, a \sin\theta \right) = 0$$

$$2\, m\, a^2\, (1 - \cos\theta)\, \ddot{\theta} + 2\, m\, a^2\, \dot{\theta}^2 \sin\theta - m\, a^2\, \dot{\theta}^2 \sin\theta - m\, g\, a \sin\theta = 0$$

Combining terms :

$$2\, m\, a^2\, (1 - \cos\theta)\, \ddot{\theta} + m\, a^2\, \dot{\theta}^2 \sin\theta - m\, g\, a \sin\theta = 0.$$

$$(1 - \cos\theta)\, \ddot{\theta} + \frac{1}{2} \sin\theta\, \dot{\theta}^2 - \frac{g \sin\theta}{2\, a} = 0$$

On the final exam, you will be asked to solve this ODE by setting u = cos (θ/2)