USING BESSEL FUNCTIONS TO SOLVE THE PROBLEM OF A VIBRATING CIRCULAR MEMBRANE WITH ASYMMETRIC INITIAL CONDITIONS

The title says it all. It might be good for you to review our solution to the vibrating circular membrane with symmetric initial conditions before diving into this.

By now, you know what the recipe calls for : write our general equation, substitute a trial solution, separate variables, then use boundary and initial conditions to determine coefficients.

So, let' s begin.

The wave equation in cylindrical coordinates is :

$$\frac{\partial^2 z}{\partial t^2} = v^2 \left(\frac{\partial^2 z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 z}{\partial \phi^2} \right)$$

where v is the velocity of the wave, z is the height of the membrane at any time t and any distance ρ from the origin.

Our trial solution is clearly :

$$z(\rho, \phi, t) = R(\rho)\,\Phi(\phi)\,T(t)$$

and after all requisite calculus and algebra we are able to separate variables to obtain these three ODES :

$$\frac{T''(t)}{T(t)} = -v^2 k^2$$

$$\frac{\Phi''(\phi)}{\Phi(\phi)} = -p^2$$

$$\frac{R''(\rho)}{R(\rho)} + \frac{1}{\rho} \frac{R'(\rho)}{R(\rho)} - \frac{p^2}{\rho^2} = -k^2$$

We know the eigenfunctions of the first two ODEs are trig functions, and the eigenfunctions of the radial equation are the appropriate Bessel J and Y functions :

$$R(\rho) = A\,J_p(k\rho) + B\,Y_p(k\rho)$$

Since we know the solution must be finite at $\rho = 0$, we can exclude the Y solutions (they go to infinity at $\rho = 0$). Also, since we know the membrane must be tied down at the edges, we have that $R(a) = 0$ where a is the radius of the membrane. Thus, we know that :

$$J_p(k a) = 0 \Rightarrow k a = \alpha_{p,n} \text{ or } k = \frac{\alpha_{p,n}}{a}$$

where the $\alpha_{p,n}$ is the n^{th}zero of the Bessel function of order p.

Now, if we combine the solutions to the three separated ODEs we get as a general solution :

$$z(\rho, \phi, t) = \sum_{p=0}^{\infty}\sum_{n=1}^{\infty} J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\left[C_{pn}\sin\left(\frac{\alpha_{p,n}}{a}v\,t\right) + D_{pn}\cos\left(\frac{\alpha_{p,n}}{a}v\,t\right)\right]\cdot \left[E_{pn}\sin(p\,\phi) + F_{pn}\cos(p\,\phi)\right] \tag{1}$$

Next, we need to make use of our initial conditions. In this case, our two initial conditions are :

$$z_0 = 0 \text{ and } \dot{z}_o = \begin{cases} s, & \rho < \rho_o,\ 0 < \phi < \pi \\ 0. & \text{otherwise} \end{cases}$$

In other words, the height of the membrane is zero everywhere at t = 0, and the vertical velocity has the value s inside a radius ρ_0 in the upper half plane. However, the vertical velocity is zero everywhere in the lower half plane, and is also zero in the upper half plane outside of ρ_0.

$z(\rho, \phi, 0) = 0$ means we can set all the D coeffcients to zero. However, since we do not have azimuthal symmetry in this case, we cannot set p = 0 as we did in class (when doing the symmetric case). Taking the time derivative of equation (1) and setting t = 0 yields :

$$\dot{z}(\rho, \phi, 0) = \sum_{p=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right) J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\left[E_{pn}\sin(p\,\phi) + F_{pn}\cos(p\,\phi)\right] \tag{2}$$

(and the C coefficient is just absored into other constants). Now, we take equation (2) and split it into two parts by extracting the p = 0 term from the double sum (you will see shortly why we do this) :

$$\dot{z}(\rho, \phi, 0) = \sum_{n=1}^{\infty}\left(\frac{\alpha_{0,n}}{a}v\right) J_0\left(\frac{\alpha_{0,n}}{a}\rho\right) F_{0,n} + \sum_{p=1}^{\infty}\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right) J_p\left(\frac{\alpha_{p,n}}{a}\rho\right) F_{pn}\right\}\cos(p\,\phi) + \sum_{p=1}^{\infty}\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right) J_p\left(\frac{\alpha_{p,n}}{a}\rho\right) E_{pn}\right\}\sin(p\,\phi) \tag{3}$$

Make sure you understand the choice of subscripts in each summation. Now, if you look carefully at eq. (3), you will see that this is merely a Fourier series in which the coefficients are written in terms of Bessel functions. The p = 0 term is simply what we have called the $a_0/2$ term, so that we have:

$$\frac{a_0}{2} = \sum_{n=1}^{\infty}\left(\frac{\alpha_{0,n}}{a}v\right) J_0\left(\frac{\alpha_{0,n}}{a}\rho\right) F_{0,n} = \frac{1}{2\pi}\int_0^{2\pi}\dot{z}(\rho, \phi, 0)\,d\phi = \begin{cases} s/2, & \rho < \rho_0 \\ 0, & \rho > \rho_0 \end{cases} \tag{4}$$

We integrate from 0 to 2π to cover all azimuthal angles; remember that the function we are writing as a Fourier series ($\dot{z}_0$) is zero on $(\pi, 2\pi)$.

Similarly, the a and b coefficients are expressed as:

$$\{a_p, b_p\} = \sum_{n=1}^{\infty}\left(\frac{\alpha_{p,n}}{a}v\right) J_p\left(\frac{\alpha_{p,n}}{a}\rho\right)\{F_{pn}, E_{pn}\} = \frac{1}{\pi}\int_0^{2\pi}\dot{z}_0(\rho, \phi)\,\{\cos(p\,\phi), \sin(p\,\theta)\}\,d\phi \tag{5}$$

The cos integral yields zero (so all the a coefficients are 0), and the b coefficients are :

$$b_p = \begin{cases} 2\,s/p, & \rho < \rho_0 \text{ for p odd} \\ 0, & \text{otherwise} \end{cases}$$

Remember that our goal is to find expressions for the coefficients E and F in equation (1). The Fourier coefficients we just found tell us that all the F_{pn} coefficients are zero for p > 0 and $E_{pn} = 0$

for even values of p. We can find expressions for $F_{0\,n}$ and the odd E_{pn} by recognizing that equations (4) and (5) are the Fourier Bessel series of the right hand sides of those equations. Using the algorithm we motivated in class for finding the coefficients of Fourier Bessel series, we can write:

$$\left(\frac{\alpha_{0,n}}{a} v\right) F_{0\,n} = \frac{2}{a^2 J_1^2(\alpha_{0,n})} \int_0^{\rho_0} \frac{s}{2} J_0\left(\frac{\alpha_{0,n}}{a} \rho\right) \rho \, d\rho \rightarrow$$

$$F_{0\,n} = \frac{\rho_0 \, s}{v \, \alpha_{0,n}^2 \, J_1^2(\alpha_{0,n})} J_1\left(\frac{\alpha_{0,n}}{a} \rho\right)$$

and

$$\left(\frac{\alpha_{p,n}}{a} v\right) E_{0\,n} = \frac{2}{a^2 J_{p+1}^2(\alpha_{p,n})} \int_0^{\rho_0} \left(\frac{2\,s}{p}\right) J_p\left(\frac{\alpha_{p,n}}{a} \rho\right) \rho \, d\rho \rightarrow$$

$$E_{\text{pn}} = 4 \frac{s}{p \, a \, v \, \alpha_{p,n} \, J_1^2(\alpha_{p,n})} \int_0^{\rho_0} J_p\left(\frac{\alpha_{p,n}}{a} \rho\right) \rho \, d\rho \qquad \text{for odd p}$$

Evaluation of these integrals is best done by computer; we substitute these expressions for the coefficients into equation (1) to produce our complete and final answer.