CCSS Mathematics Expectation:
Problem situations that are language-rich and require multiple steps to decipher text for relevant phrases and specific use of language structures, vocabulary, relationships, concepts, and goals

CCSS Mathematics Expectation:
Concepts represented in multiple ways and require translation between and among words, numbers, tables, diagrams, and symbols

CCSS Mathematics Expectation:
Procedures constitute a special narrative (i.e., step-by-step actions lead reliably to a result) that support the determination of relevant ideas and the reasonableness of an answer
1. Mathematical language is more than vocabulary; learning language and content involves expanding *linguistic repertoires* to engage in a wide variety of situations, with a wide variety of concepts, for a wide variety of purposes.

2. English learners (ELs) bring rich linguistic backgrounds and repertoires into the classroom, including those in their native language or linguistic varieties.

3. ELs have rich background knowledge and experiences, which must be the basis for all teaching and learning; instruction must consider *and expand* what ELs bring to the classroom.

4. Teachers must provide ELs with abundant and diverse opportunities to engage with the four domains of language while attending to the affective filter, making students feel comfortable to take risks with their language and linguistic repertoires.

5. Since language and cognition develop simultaneously, teachers need to support and scaffold language-rich mathematics discussions. ELs learn to do things with language when they are engaged in meaningful content-based activities that engage and challenge them.

6. Mathematical practice and literacy requires multiple representations of concepts and ideas. Teaching and learning of mathematics for ELs requires use of visual and hands-on resources, such as charts, graphics, manipulatives, and realia.

7. In order to develop deep understanding of mathematical concepts, ELs need access to such concepts, complex texts, and academic conversations, along with support and extended time to deeply engage and wrestle with them.
CCSS require mathematical discourse centered on learning strategies. Teachers must understand and support the linguistic development of students by modeling and prompting language to describe strategies, patterns, generalizations, and representations.
(A) Learning Strategies Defined

Learning strategies are techniques that facilitate the process of understanding, retaining, and applying knowledge. Needing guidance to overcome the challenge of learning a new language while trying to use that language as a means to learn content, ELLs benefit from explicit teaching of learning strategies, including metacognitive, cognitive, memory, social, and compensation strategies. Review the list of examples of learning strategies. Consider and discuss: (1) How do you currently teach learning strategies in math? (2) How do your ELLs respond?
(B) Actively Teach Learning Strategies

 Particularly for ELLs, strategy training cannot be subtle but taught explicitly and overtly. This allows for simultaneous cognitive and linguistic development, as ELLs develop learning strategies and mathematical discourse. Strategy teaching is a five-step process that includes thinking aloud, modeling, and prompting. (1) Review the description of the five steps to teach learning strategies. (2) Select a strategy from the list that relates to a current math topic in your classroom. (3) Discuss how you would teach this strategy to support language.
Figure 9.3 The Traditional Model for Reading a Textbook

1. Read the text.
2. Answer the questions.
3. Discuss in class.
4. Apply to real life.

Figure 9.4 A New Model for Reading a Textbook: Reading in Reverse

4. Read the text.
3. Answer the questions.
2. Discuss in class.
1. Apply to real life.
1. *Introduce* it and label it as a new strategy.
2. *Identify* it by name and explain its use.
3. *Demonstrate* how to use it.
4. Give students time and opportunity for *practice*.
5. *Discuss* effectiveness and application to other tasks.

**Step 1: Make language substitutions.**
In pairs or small groups, students look for words or phrases that can be eliminated or replaced with more simple language. They use a bilingual dictionary or a student dictionary as needed.

**Step 2: Determine the information presented.**
Students, still in pairs or groups, reread the now-simplified wording of the problem. They search out and write down all information given in the problem.

**Step 3: Determine the information needed for solution.**
Students now look for words that offer clues to the information needed in the solution. They first eliminate extraneous words and information, and then write out the words and phrases that tell how to process the information in the problem.

**Step 4: Determine the process needed for solution.**
Using the words or phrases from Step 3, students figure out the process needed to find the solution.

**Step 5: Solve the problem.**
Students perform the necessary computations and compare results.
Metacognitive Strategies

Metacognitive strategies are those that involve thinking about learning. These can be divided into two subtypes of techniques: those that deal with organizing and planning for learning, and those that deal with self-monitoring and self-evaluating learning.

Examples of Metacognitive Strategies Dealing with Organizing and Planning for Learning

- Using a homework notebook to write down all assignments.
- Keeping a calendar/organizer to write down long-term assignments.
- Dividing long-term assignments into shorter segments and tasks.
- Setting deadlines for completion of each segment or task.

- Determining the most appropriate and efficient strategies to learn specific content.
- Planning how to study for a test

Examples of Metacognitive Strategies Dealing with Self-Monitoring and Self-Evaluation of Learning

- Recognizing your own knowledge gaps or weaknesses
- Discovering strategies that work best for you (and those that don't)
- Training yourself to monitor your own progress in learning—to awaken the little voice in your head that asks, "How am I doing?" and "Am I understanding this?"
- Checking your progress frequently by responding to that little voice
- Recognizing the need to find a new strategy if the one you're using isn't working
Cognitive Strategies

Cognitive strategies are those that involve any type of *practice* activity. These are techniques that promote deeper understanding, better retention, and/or increased ability to apply new knowledge. The techniques that fall into this category are familiar to successful learners and are used on a regular basis.

Examples of Cognitive Strategies

- Making specific connections between new and old learning
- Making specific connections between English and the student's native language
- Highlighting important information while reading
- Dividing a large body of information into smaller units
- Note taking (even in student's native language)
- Condensing notes to study for a test
- Making and using flash cards to test yourself
- Making visual associations to aid in retention
- Creating graphic organizers, maps, charts, diagrams, timelines, and flowcharts to organize information
- Making categories and classifications

Cognitive strategies form the core of learning techniques. They fall into the general category that students call *studying*. Cognitive strategies that are creative, interesting, even gamelike in nature, put a positive spin on studying, making it more motivating and productive to all students.
Memory Strategies

Memory strategies consist of any technique that aids *rote recitation* of learned material. Memory strategies are devised simply to recall elements without any attempt to understand the material more completely. They are a subtype of cognitive strategies because their purpose is to trigger the recall of specific groups of items, concepts, or ideas that have been learned through other cognitive techniques.

A simple example of a memory strategy is the *Alphabet Song* learned by young children often long before they have any concept of letters. It is strictly a rote memory device—a *mnemonic*—and indeed, a very effective one. Later, as children learn to recognize and write their letters, they use the *Alphabet Song* to remind themselves of alphabetic order.

Mnemonics in academia are created, often by individual students, to help remember rules, key words, lists, and categories. Memory strategies such as poems, songs, acronyms, sentences (the first letter of each word in the sentence is the same as an item in an ordered list), and word patterns are very effective in triggering the recall of much larger bodies of information that have been learned through other cognitive approaches. ELLs can even use their native languages to create their own memory devices.

Examples of Familiar Mnemonics

- The “I before E” poem to recall spelling rules
- The poem and the “knuckle technique” for remembering which months of the year have less than 30 days (see Figure 4.1)
- The made-up word to remember the color spectrum in which the letters recall the colors in their correct order (see Figure 4.2)
- The silly sentence to remember the notes on a treble staff (see Figure 4.2)
- The “tricks” of the multiplication table (see Figure 4.3)

As students recognize the efficiency of mnemonics as a recall tool and feel comfortable using them, you can assign the creation of a mnemonic as a creative homework assignment, as described in Figure 4.4.
Social Strategies

Social strategies are of two types. In the first type, language learners attempt to learn English by interacting with the environment. They expand their vocabularies by listening as English is being spoken and by attending to written English as it appears around them.

The second type of social strategy is more closely related to the classroom. Here language learners work with one or more other students to learn information or to complete a task. Group work and cooperative learning are social learning strategies. Because social strategies are, as the name states, social, they often feel less like practice and occasionally even like fun.

Examples of Social Strategies

- Working in class in pairs or small groups to clarify content, solve problems, and complete projects
- Playing teacher-made or professionally designed games to sharpen skills
- Doing homework with a friend
- Studying with a partner for a test
- Watching select television programs
- Observing peers to learn more about culture and language
- Asking questions and making requests (see Figure 4.5)

Compensation Strategies

Compensation strategies are techniques used to make up for something that is either unknown or not immediately accessible from memory. Proficient speakers of English regularly use this type of strategy in conversations when they use words or phrases such as *whatchamacallit*, the *thingamajiggy*, or just plain *that thing—you know what I mean* to replace the language they are searching for but cannot find at the moment.

Examples of Compensation Strategies

- Stalling for time while we think of an appropriate response
- Making an educated guess that extends and generalizes what we know to what we don’t know
- Using a circumlocution, a substitute phrase that “talks around” the word we don’t know or “writes around” the word we can’t spell
MATHEMATICAL BACKGROUND KNOWLEDGE

Instruction grounded in Math CCSS must integrate and build on the background knowledge that students bring from home, community, and school.
Luis Moll describes funds of knowledge to be “a positive and realistic view of households as containing ample cultural and cognitive resources with great potential for utility for classrooms” (1992, p. 134). Families engage in rich numeracy practices, such as cooking, making and selling items, bartering, purchasing commodities, and telling time. Using the list of examples of funds of knowledge, think about the math practices your students engage in at home. How can you glean this background knowledge and incorporate into instruction?
ELLs may be learning English, but they bring many linguistic resources to the classroom. Students’ native languages are assets in math instruction for cognitive and linguistic development. (1) Cognitive: Read brain-based research summary on cognitive processing differences based on native language. How does this research inform your thinking about ELLs in your classroom? (2) Linguistic: Browse the Spanish-English math cognates. How can you use students’ native languages to develop English language proficiency and build vocabulary?
Students come into your classroom with background knowledge and experiences from academic settings, whether from prior schooling in their native country or in other classrooms in the U.S. (1) Consider affective factors (e.g., self-esteem, motivation, interest) from prior math experiences in school. (2) Consider cognitive factors from prior math instruction, specifically identifying students’ personal preferred strategies for making meaning of math problems. How can you collect and utilize affective and cognitive background knowledge? Browse strategies to access background knowledge.
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Table 1
A Sample of Household Funds of Knowledge
## Math Cognates

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# Math Cognates

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# Math Cognates

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<td>Plane figure</td>
<td>la figura plana</td>
</tr>
<tr>
<td>Polygon</td>
<td>el polígono</td>
</tr>
<tr>
<td>Position</td>
<td>la posición</td>
</tr>
<tr>
<td>Prediction</td>
<td>la predicción</td>
</tr>
<tr>
<td>Price</td>
<td>El precio (noun only; in the sense of “cost”)</td>
</tr>
<tr>
<td>Prime number</td>
<td>el número primo</td>
</tr>
<tr>
<td>Prism</td>
<td>el prisma</td>
</tr>
<tr>
<td>Probability</td>
<td>la probabilidad</td>
</tr>
<tr>
<td>Problem</td>
<td>el problema</td>
</tr>
<tr>
<td>Product</td>
<td>el producto</td>
</tr>
<tr>
<td>Pyramid</td>
<td>la pirámide</td>
</tr>
<tr>
<td>Quadrant</td>
<td>el cuadrante</td>
</tr>
<tr>
<td>Quadrilateral</td>
<td>el cuadriláteral (noun): cuadriláteral (adj.)</td>
</tr>
<tr>
<td>Quotient</td>
<td>el cociente</td>
</tr>
<tr>
<td>Radii</td>
<td>los radios</td>
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<td>el radio</td>
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<tr>
<td>Range</td>
<td>el rango</td>
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<tr>
<td>Reciprocal</td>
<td>recíproco/a (adj. only); not a cогnate for noun</td>
</tr>
<tr>
<td>Rectangle</td>
<td>el rectángulo</td>
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<tr>
<td>Rectangular prism</td>
<td>el prismo rectángulo</td>
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<tr>
<td>Vocabulary Word</td>
<td>Cognate</td>
</tr>
<tr>
<td>------------------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>Rectangular pyramid</td>
<td>la pirámide rectangular</td>
</tr>
<tr>
<td>Reflection</td>
<td>la reflección</td>
</tr>
<tr>
<td>Regroup</td>
<td>reagrupar</td>
</tr>
<tr>
<td>Represented</td>
<td>representado/a (adj.)</td>
</tr>
<tr>
<td>Represents</td>
<td>representa</td>
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<td>Results</td>
<td>los resultados</td>
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<td>Rhombus</td>
<td>el rombo</td>
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<td>la rotación</td>
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<td>Scalene</td>
<td>escaleno/a</td>
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<td>el triángulo scaleno</td>
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<tr>
<td>Seconds</td>
<td>los segundos</td>
</tr>
<tr>
<td>Segment</td>
<td>el segmento</td>
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<td>Sign</td>
<td>el signo</td>
</tr>
<tr>
<td>Simplify</td>
<td>simplificar</td>
</tr>
<tr>
<td>Solution</td>
<td>la solución</td>
</tr>
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<td>Standard form</td>
<td>la forma estándar</td>
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<tr>
<td>Sum</td>
<td>la suma</td>
</tr>
<tr>
<td>Symmetry</td>
<td>la simetría</td>
</tr>
<tr>
<td>Table</td>
<td>la tabla</td>
</tr>
<tr>
<td>Temperature</td>
<td>la temperatura</td>
</tr>
<tr>
<td>Term</td>
<td>el término</td>
</tr>
<tr>
<td>Total</td>
<td>el total</td>
</tr>
<tr>
<td>Transformation</td>
<td>la transformación</td>
</tr>
<tr>
<td>Trapezoid</td>
<td>el trapezio</td>
</tr>
<tr>
<td>Triangle</td>
<td>el triángulo</td>
</tr>
<tr>
<td>Triangular pyramid</td>
<td>la pirámide triangular</td>
</tr>
<tr>
<td>Value</td>
<td>el valor</td>
</tr>
<tr>
<td>Variable</td>
<td>la variable (noun): variable (adj.)</td>
</tr>
<tr>
<td>Vertex</td>
<td>el vértice</td>
</tr>
<tr>
<td>Vertical</td>
<td>vertical</td>
</tr>
<tr>
<td>Vertices</td>
<td>los vértices</td>
</tr>
<tr>
<td>Volume</td>
<td>el volumen</td>
</tr>
<tr>
<td>Yard</td>
<td>la yarda</td>
</tr>
<tr>
<td>Zero</td>
<td>el cero</td>
</tr>
</tbody>
</table>
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Carousel Brainstorming

**Purpose:** To activate students' prior knowledge of a topic or topics through movement and conversation.

**Description:** While Carousel Brainstorming, students will rotate around the classroom in small groups, stopping at various stations for a designated amount of time. While at each station, students will activate their prior knowledge of different topics or different aspects of a single topic through conversation with peers. Ideas shared will be posted at each station for all groups to read. Through movement and conversation, prior knowledge will be activated, providing scaffolding for new information to be learned in the proceeding lesson activity.

**Procedure:**
1. Generate X number of questions for your topic of study and write each question on a separate piece of poster board or chart paper. (Note: The number of questions should reflect the number of groups you intend to use during this activity.) Post questions sheets around your classroom.
2. Divide your students into groups of 5 or less. For example, in a classroom of 30 students, you would divide your class into 6 groups of five that will rotate around the room during this activity.
3. Direct each group to stand in front of a homebase question station. Give each group a colored marker for writing their ideas at the question stations. It is advisable to use a different color for tracking each group.
4. Inform groups that they will have X number of minutes to brainstorm and write ideas at each question station. Usually 2-3 minutes is sufficient. When time is called, groups will rotate to the next station in clockwise order. Numbering the stations will make this easy for students to track. Group 1 would rotate to question station 2; Group 2 would rotate to question station 3 and so on.
5. Using a stopwatch or other timer, begin the group rotation. Continue until each group reaches their last question station.
6. Before leaving the final question station, have each group select the top 3 ideas from their station to share with the entire class.


---

**Sample Carousel Brainstorming for Databases**

**Question Stations:**
1. What is a database used for?
2. What do you see when viewing a database?
3. What are examples of databases that we use in everyday life?
4. What fields (categories) of information would you place in a database of your friends?
5. What fields (categories) of information would you place in a database of European countries?
6. What types of information do not necessarily belong in a database?

**Sample Carousel Brainstorming for Webpage Evaluation**

1. What should a good webpage look like?
2. What type of information should you see on a good webpage?
3. What information would you expect to find on a webpage about European countries?
4. What information would you expect to find on a webpage about biomes?
5. What are some examples of things NOT to put on your webpage?
6. If you could design your ideal webpage, what are some features you would include?
Sample Room Layout for Carousel Brainstorming

- Question Station #1
- Question Station #2
- Question Station #3
- Question Station #4
- Question Station #5
- Question Station #6
Two Minute Talks

**Purpose:** To activate prior knowledge and focus student learning on the topic about to be addressed.

**Description:** During Two Minute Talks, students will share with a partner by brainstorming everything they already know (prior knowledge) about a skill, topic, or concept. In doing so, they are establishing a foundation of knowledge in preparation for learning new information about the skill, topic, or concept.

**Procedure:**
1. Group students into pairs.
2. Inform students that they will each be talking about topic X for two minutes. They will need to select which student will begin first. An easy way to do this is to say something like: "Find out whose birthday comes first in a calendar year." Then tell students that, "That person gets to go second!"
3. Using a stop watch or other timing device, tell students to begin talking.
4. At two minutes, instruct students to switch. At this point, the other partner begins talking. It is okay for the second person to repeat some of the things the first person said. However, they are encouraged to try and think of new information to share.
5. Have a few groups share some of their responses with the entire class when the activity is done.

**Sample Two Minute Topics:**
- What are the benefits of using the internet?
- What would happen to schools if all the computers disappeared overnight?
- Name as many topics for databases that you can think of.
- How would you use a PowerPoint slideshow to convince your parents to increase your allowance?
- Name all of the things you can do in a word processing program.

*from Instructional Strategies for Engaging Learners*  
*Guilford County Schools TF, 2002*
Think-Pair-Share

**Purpose:** To engage students in about their prior knowledge of a topic.

**Description:** During this activity, students will have individual time to think about a question related to the topic of study. They will then pair up with a partner to share their thoughts. Finally, the pairs will select one major idea to share with the entire class.

**Procedure:**
1. Generate a higher-level question related to the topic you are about to study.
2. Group students into pairs.
3. Pass out a Think-Pair-Share worksheet to each student.
4. Give students 5 minutes to write down their individual thoughts in the "Think" section of the worksheet.
5. Then, in pairs, have groups share their individual thoughts. Pairs should summarize their common thoughts in the "Pair" section of their worksheet.
6. Finally, pairs choose one major idea to share with the entire class. This should be written in the "Share" section of their worksheet.

**Sample Think-Pair-Share Questions:**
- What are the important elements of a multimedia slideshow presentation?
- How would you evaluate the quality of a webpage?
- What jobs might require the use of a spreadsheet?
- What are some of the things you need to think about before building a database?
- What are the advantages and disadvantages of using the internet for research?
- Should everyone have access to the Internet?


*from Instructional Strategies for Engaging Learners*

*Guilford County Schools TF, 2002*

*Return to Activating Strategies*
Sample Think-Pair-Share for PowerPoint

Think

Think about both of the PowerPoint presentations you have just viewed. Which presentation did you prefer? Explain why in the space below:

_____________________________________________________________________________________
_____________________________________________________________________________________
_____________________________________________________________________________________
_____________________________________________________________________________________

Pair

Pair up with a partner. Start a discussion with your partner by asking him/her which presentation they preferred. Ask your partner to explain in detail why they preferred one PowerPoint presentation to the other. Combine your ideas and summarize your discussion below:

_____________________________________________________________________________________
_____________________________________________________________________________________
_____________________________________________________________________________________
_____________________________________________________________________________________

Share

Share with the whole class the most important points from your "Paired" discussion. To prepare for sharing, list below the three most important points you would like to share with the entire class:

1.  

2.  

3.  

Think-Pair-Share

My question:

Think

During the next 5 minutes, think about your answer to the question above. Write your response on the lines below:

Pair

Now, pair up with your partner to exchange ideas? What ideas did you have in common? Write those ideas below:

Share

Using your "Pair" ideas, decide upon one major idea to share with the whole class. Write that major idea below:
Talking Drawings

**Purpose:** To activate and evaluate student knowledge of a topic.

**Description:** In this activity, students will activate prior knowledge by creating a graphic representation of a topic before the lesson. After engaging in learning about that topic, students will re-evaluate their prior knowledge by drawing a second depiction of their topic. They will then summarize what the different drawing say to them about what they learned.

**Procedure:**

1. Ask students to close their eyes and think about topic X. Using the Talking Drawings worksheet, have students draw a picture what they saw while they were thinking about topic X.

2. Teach cognitive portion of your lesson.

3. At the end of the lesson, ask students to elaborate upon their initial drawing by creating a new drawing that incorporates what they learned about topic X during the lesson.

4. Have students share their before and after drawings with a partner. Students should discuss the differences between the two depictions of topic X.

5. Finally, have students respond in writing at the bottom of their Talking Drawings worksheet. What do the two drawings tell them about what they learned during the lesson?


*from Instructional Strategies for Engaging Learners*  
*Guilford County Schools TF, 2002*
Talking Drawings

1. Close your eyes and think about ______________________________. Now, open your eyes and draw what you saw.

2. Now that you have learned more about __________________________, draw a second picture to show what you learned.

3. In the space below, tell what you have changed about your before and after pictures. Explain why you made those changes.

___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
The First Word

**Purpose:** To activate students’ prior knowledge of a concept, idea, or skill

**Description:** The First Word is a variation on traditional acronyms. By going through the process of analyzing words and creating related sentences, students will gain a deeper understanding of the meaning.

**Procedure:**
1. Assign students the name of an object, a topic, or key concept to write vertically down the side of a page.
2. Working in small groups or on their own, students should generate a short phrase or sentence that begins with each letter of the vertical work and offers important information or key characteristics about the topic.
3. Students can illustrate their "First Words" for posting around the classroom. Sharing "First Words" will allow students to identify important concepts that may have been left out of their own work.

---

**Sample First Word:**

Sun is the star at the center of the solar system
Orbits are the paths that planets take around the Sun
Lunar eclipses occur when the Moon gets blocked by the Earth
Asteroids are big rocks that orbit the Sun
Rings-- the planet Saturn has them
Saturn is the sixth planet from the Sun
You can see some planets with your naked eye
Some other planets are: Earth, Venus, Mars, Jupiter, Pluto, and Neptune
The Earth is the only planet with life on it
Every year, the Earth orbits the Sun once
Mercury is the planet closest to the Sun


from *Instructional Strategies for Engaging Learners*

*Guilford County Schools TF, 2002*
The First Word
Walk Around Survey

**Purpose:** To activate students’ prior knowledge through conversation and movement

**Description:** Walk Around Survey can be used as an activating or summarizing strategy. In this activity, students are given a topic of study and asked to move around the room for the purpose of conversing with other students. During these conversations, students will share what they know of the topic and discover what others have learned.

**Procedure:**
1. Assign a topic for the Walk Around Survey.
2. Pass out a survey form to each student in the class.
3. Allow students an allotted amount of time to survey three classmates (informers) on the given topic.
4. When students are completing the survey form, the soliciting student should write the name of the informer on his/her worksheet in the left-hand column. He/she will then record three facts from the student informer on the worksheet in the three empty blocks. He/she will then move on to find a second and third informing student to complete the survey worksheet.
5. Have students return to their seats and complete the Survey Summary.

**Hint:** This activity can be used as either an activating or summarizing strategy. It can be done in the classroom or, even better, outside on a nice day.

**Sample Walk Around Survey Topics:**
1. What can you do to become a responsible user of the Internet?
2. If you were creating a database about X, what fields would you most likely include?
3. Name ways in which spreadsheets are used in the workplace.
4. How has the Internet changed the way we communicate and interact with others?


*from Instructional Strategies for Engaging Learners*
*Guilford County Schools TF, 2002*
## Walk Around Survey

**Topic:** ________________________________

<table>
<thead>
<tr>
<th>Fact #1</th>
<th>Fact #2</th>
<th>Fact #3</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Informer #1" /></td>
<td><img src="image2" alt="Informer" /></td>
<td><img src="image3" alt="Informer" /></td>
</tr>
</tbody>
</table>

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14
Walk Around Survey Summary

Briefly summarize what you have learned from your student informers:

___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________

Upon which topics do you still need more information?

___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________

What questions do you have?

___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
___________________________________________________________________
Three Step Interview

**Purpose:** To engage students in conversation for the purpose of analyzing and synthesizing new information.

**Description:** The Three Step Interview is a cooperative structure that helps students personalize their learning and listen to and appreciate the ideas and thinking of others. Active listening and paraphrasing by the interviewer develops understanding and empathy for the thinking of the interviewee.

**Procedure:**
1. Students work in pairs. One is the interviewer, the other is the interviewee. The interviewer listens actively to the comments and thoughts of the interviewee, paraphrasing key points and significant details.
2. Student pairs reverse roles, repeating the interview process.
3. Each pair then joins another pair to form groups of four. Students introduce their pair partner and share what the partner had to say about the topic at hand.

**Sample Three Step Interview Topics:**
1. Present a very challenging filter/sort combination problem to the students. Allow them to use the interview to discuss possible solutions.
2. Present students with an ethical situation related to privacy and the internet. Allow students to use the interview as a means of discussing the different components of the issues at hand.
3. Provide students a short (4-5 words) list of vocabulary to be reviewed. In the interview, they are to explain the definitions and applications of the words. By regrouping with the other interview pair, appropriate student use of vocabulary will be reinforced.


*from Instructional Strategies for Engaging Learners*

*Guilford County Schools TF, 2002*
In the Hot Seat

**Purpose:** To motivate student learning

**Description:** In this activity, several students will be asked to sit in the "Hot Seat" and answer questions related to the topic of study.

**Procedure:**

1. Prior to the beginning of class, the teacher will prepare questions related to the topic of study and write them on sticky notes. Four to five questions are usually enough.
2. Place the sticky notes underneath student desks/chairs so that they are hidden from view.
3. At the start of the class, inform students that several of them are sitting on "Hot Seats" and will be asked to answer questions related to the topic of study for the day.
4. Have students check their desks/chairs for the strategically placed sticky notes.
5. Students who have questions on sticky notes will then take turns reading the question and attempting to provide an answer. Due to the nature of this motivational activity, these should be questions that students are able to answer.

---

**Sample Hot Seat Questions:**

**Internet:**

1. What is your favorite search engine and why?
2. When was the last time you used the internet to complete a classroom assignment?
3. If you had to recommend a website to a friend, which one would you pick and why?
4. What do you think would be the impact if the Internet was gone tomorrow?
5. Do you think that students should be allowed to use the Internet unsupervised? Why or why not?

*from Instructional Strategies for Engaging Learners*

*Guilford County Schools TF, 2002*
THIEVES: A Strategy for Previewing Textbooks

This activity will help students with comprehension by allowing them to preview the text structure in an organized manner. This pre-reading strategy will allow students to “steal” information before they actually begin reading the chapter. Students will survey the text in the following manner:

**Title** – Students sometimes skip the title, but it provides valuable information by establishing the topic and the context of the chapter. If the text is written in chronological order, the title may indicate where the chapter would fit on a timeline. Some questions that the student may ask while looking at the title include:

- What do I already know about this topic?
- How does it connect to the previous chapter?
- How can I turn this title into a question to focus my reading?

**Headings** – Headings indicate the important sections of the chapter. They help students identify the specific topics covered. Students can turn the headings into questions to create a more focused look at information covered in the chapter. Some questions that the student may ask while looking at the headings include:

- How does this heading let me know what I will be reading about?
- What topic will be discussed in the paragraphs below this heading?
- How can I turn this heading into a question that can be answered when I read this section?

**Introduction** – The introduction provides an overview of the chapter. It may come after the title and before the first heading. Sometimes the goals and objectives of the chapter are stated in the introduction. Some questions that students may ask when previewing the introduction include:

- Is the introduction marked or do I have to locate it?
- Does the first paragraph introduce the chapter?
- What important information will I find in the introduction?
- Do I already know anything about this?

**Every first sentence in a paragraph** – First sentences are often the topic sentences of the paragraph, and by reading these a student can get an idea of the information that will be contained in the chapter.
**Visuals and Vocabulary** – Students should look at all pictures, charts, tables, maps and graphs contained in the chapter. They need to read the captions and labels on each. This enables students to learn a little about the topic before they begin to read. Some questions that students may ask about the visuals include:

- How do these visuals relate to the content of this chapter?
- What can I learn from them?
- How do the captions help me understand the visual?

Vocabulary unlocks the meaning of the content. Students need to understand vocabulary in order to comprehend the text. Vocabulary may or may not be identified as key words. It might be highlighted or italicized in the text. Some questions that students may ask about the vocabulary include:

- Is there a list of key words and are they defined in the glossary?
- Are there important words in boldface or italics?
- Do I know the important words?
- Are there other words I don’t know?

**End-of-Chapter Questions** – These questions indicate important points and concepts from the chapter. Just reading these questions will help students target information that is important in the text and establish a purpose for reading. Some questions that students may ask about the end-of-chapter questions include:

- What do these questions ask?
- What information will be important in this chapter?
- How do I locate this information in the text?

**Summary** – Many texts contain a summary at the end of the chapter. Students can read the summary to activate prior knowledge and give them an idea of the important concepts contained in the chapter.

THIEVES was created by Suzanne Liff Manz, an educational therapist and instructor at Nassau Community College in Garden City, NY. It was published in *The Reading Teacher* Volume 55 Number 5 in February, 2002.
Kinesthetic THIEVES

Because you will have many students who are kinesthetic learners, here is a way for them to learn the THIEVES technique through movements.

TITLE – Explain that a king or queen has a title and they wear a crown. Make the crown by circling the fingers of one hand and placing in on the top of the head.

HEADING – Do the *Home Alone* face that students may remember from the movie. Place both hands on the cheeks of the face and open the mouth wide.

INTRODUCTION – Explain to students that usually when we are introduced to someone, we shake his or her hand. For this movement, extend the right hand and act as if you are greeting someone.

EVERY FIRST SENTENCE – We read from left to right. Extend the right hand to the left side of the body and bring it back to the right as if you were reading word by word and pointing to them.

VISUALS AND VOCABULARY – Form a V with two fingers on each hand and place them under each eye. Remind students that these are two things they must “look” at in the text.

END OF CHAPTER QUESTIONS – This usually gets a giggle. Place one hand on your hip near your behind.

SUMMARY – Explain that a summary gives an overview of the whole thing. Make a huge circle with both hands.

Make sure that students say the steps in the THIEVES technique as they are doing the motions. The more repetition students have with this the more familiar they will become, and the more easily they will be able to use it.
T.H.I.E.V.E.S. Questions

Students and parents, here is a great strategy to preview chapters of any textbook. It is known as T.H.I.E.V.E.S., an acronym for the steps of the strategy. After a few times of practice, you will find this strategy easy, and very effective in improving your comprehension of what you read.

T……. TITLE
What is the title?
What do I already know about this topic?
What does this topic have to do with the preceding chapter?
Does the title express a point of view?
What do I think I will be reading about?

H…….HEADINGS/SUBHEADINGS
What does this heading tell me I will be reading about?
What is the topic of the paragraph beneath it?
How can I turn this heading into a question that is likely to be answered in the text?

I…….INTRODUCTION
Is there an opening paragraph, perhaps italicized?
Does the first paragraph introduce the chapter?
What does the introduction tell me I will be reading about?

E…….EVERY FIRST SENTENCE IN A PARAGRAPH
What do I think this chapter is going to be about, based on the first sentence in each paragraph?

V…….VISUALS AND VOCABULARY
Does the chapter include photographs, drawings, maps, charts, or graphs?
What can I learn from the visuals in a chapter?
How do captions help me better understand the meaning?
Is there a list of key vocabulary terms and definitions?
Are there important words in boldface type throughout the chapter?
Do I know what the bold-faced words mean?
Can I tell the meaning of the boldfaced words from the sentences in which they are embedded?

E…….END-OF-CHAPTER QUESTIONS
What do the questions ask?
What information do I learn from the questions?
Let me keep in mind the end-of-chapter questions so that I may annotate my text where pertinent information is located.

S…….SUMMARY
What do I understand and recall about the topics covered in the summary?
Using Picture Books

Picture books have been used in the primary grades for decades, but they are a quick and convenient way to help older students activate their prior knowledge. There is a new focus on picture books that deliver difficult content in simple language. Picture books are a great model for student writing, as they contain vivid language and a variety of text structures. The beauty of using picture books in the upper grades, middle and high school, is that they can be read in a few minutes and provide students with information connected to the concept or skill being introduced.

Picture Books for Older Students

Here is a bibliography of picture books for secondary teachers. They are grouped by subject area for convenience.

**Social Studies:**


**Science:**


**Math:**


**English:**


**Character Building:**


**Multicultural Books:**


**Just Great Books:**


**Professional Books:**

When students are given the opportunity to brainstorm ideas without criticism, to discuss opinions, to debate controversial issues, and to answer questions...wonderful things can happen that naturally improve comprehension and higher order thinking.”

Marcia Tate, 2004

OVERVIEW

A Carousel Brainstorm is a variation of the Walkabout Review process. Chart paper containing several statements or issues for student consideration are posted at strategic locations around the classroom. Groups of students brainstorm at one station and then rotate to the next position where they add additional comments. As new thoughts and ideas emerge, the list grows. When the carousel “stops” the original team prepares a summary and then presents the large group’s findings.

The Carousel Brainstorm provides an opportunity to use the group’s collective prior knowledge to further individual student understanding. It is an active, student-centered method for generating and sharing large amounts of data. Because the process is somewhat anonymous, even the most reluctant learners are motivated to participate.

IMPLEMENTING THIS ACTIVITY

1. Clearly state the problems, questions, or issue statements to be explored on large pieces of chart paper.
2. Give instructions for completing the Carousel Brainstorm.
3. Separate the class into equal groups based on the number of problems, questions, or issue statements (e.g., 5 problems, 5 groups).
4. Distribute sticky notes and markers to each group.
5. Groups brainstorm ideas on the sticky notes, coming up with as much information as possible.
6. Use one sticky note per thought or idea. Generally short phrases or sentences work best.
7. Place the sticky notes randomly on the chart paper under the problem, question, or issue statement.
8. To encourage creative and open thinking, groups consider all ideas without evaluating their accuracy or relative importance.
9. After a specified time, groups move to the next station in a clockwise pattern.
10. During each round of the Carousel Brainstorm new ideas are added to expand the information base.
11. When groups rotate back to their original positions, the data is collated on a new piece of chart paper.
12. Each group’s spokesperson summarizes the findings to the larger group.
13. If students maintain a notebook, have them write a summary reflection that captures the essence of what they discovered during the Carousel Brainstorm.
ASSESSING THIS ACTIVITY

In brainstorming activities such as this, student performance is not typically assessed. Teachers may want to generally evaluate a student’s level of participation or review the final reflections.

MANAGING THIS ACTIVITY

1. To support quality discussion and analysis, use a maximum of five - six problems, questions or issue statements.
2. Create problems questions or issue statements that stimulate discussion. Overly difficult questions will frustrate students and inhibit thoughtful generation of ideas.
3. Use different color sticky notes for different stations.

CONTENT AREA APPLICATIONS

**English:** In reviewing a book that the class has been reading, the teacher posts topics and has students enter information that they have gleaned about the book. Topics could include main characters, themes, symbolism, setting, critical events, etc.

**History:** In a unit that targets American leadership, names of several presidents are posted. Students brainstorm information that they already know about these individuals. Posters are displayed prominently through the remainder of the unit. Information is added or deleted from the lists as the unit unfolds.

**Biology:** The teacher prepares poster lists of the principal cell structures. Students complete the Carousel by adding pictures, drawings, information, or questions about each item.

REFERENCE

North Central Regional Educational Laboratory. (1999). Blueprints CD-ROM.
CCSS requires language-rich mathematical dialogue among students, which typically includes both social and academic language. Teachers must scaffold and support based on the language proficiency levels of students.
(A) Scaffolding for Oral Math Reasoning
Instruction should focus on uncovering, hearing, and supporting students’ math reasoning, not on accuracy in using language. Teachers should focus on meaning, no matter the type of language students may use. After students have ample time to engage in math practices orally and in writing, instruction can then consider how to move students toward accuracy. Consider and discuss: (1) How can you engage students in math thinking and dialog to uncover mathematical ideas in developing language? (2) How can you engage and support students’ participation in math discussion and practices?
(B) Scaffolding for Academic Vocabulary

When asking students to produce mathematical language (i.e., speaking, writing) individually or in small groups, teachers can provide scaffolds to support production of academic language. These scaffolds should correspond to ELLs’ level of language proficiency, such as word banks for beginners or sentence starters for intermediate ELLs. Discuss: (1) How can academic language scaffolds support ELLs without lowering expectations? (2) How can you support academic discourse (not simplified language) in small groups?
(C) Scaffolding for Reading Comprehension

To support the reading of textbooks and other complex texts in mathematics, teachers can utilize the “reading in reverse” scaffolding strategy to review concepts to prepare students to mentally organize a forthcoming reading. Look at the figures that compare traditional and reverse models for reading textbooks. Consider and discuss the approach: (1) How might reading in reverse support students’ comprehension of math textbooks? (2) How does reading in reverse contribute to language-rich math instruction?
Addition and Subtraction Books

Merriam, Eve. *12 Ways to Get to 11*. Introduces familiar situations to children that aid in the concept that a single number can be made up of many combinations of other numbers.

Owen, Annie. *Annie's One to Ten*. Not only is *Annie's One to Ten* a great counting book, it is a book that shows all the combinations of numbers that add up to ten. Children, teachers, and parents will enjoy the colorful and entertaining way this book presents addition facts.

Counting Books

Anno, Mitsumasa. *Anno's Counting Book*. (suitable For Grades 1-6) Numbers, sets, and groups take place in a real setting in this book that depicts the growth of a community.

Carter, David A. *How Many Bugs in a Box?* Curious readers anxiously open each box in this book to discover what kind of zany bug waits to be counted!

Crowther, Robert. *The Most Amazing Hide-and-Seek Counting Book*. One hundred charming creatures and other things can be discovered and counted by sliding, pulling, and lifting the pages of this highly interactive counting book.

Lindbergh, Reeve. *The Midnight Farm*. At bedtime, a mother walks her young child around the farm to see the animals in the darkness. In this gentle counting book, the mother helps her child understand that the darkness of night is a comfortable and safe place to be.

Division Books

Hutchins, Pat. *The Doorbell Rang*. (Suitable for Grades 3-5) As a mother offers her children a plate of cookies to share, the doorbell rings. Two neighboring children join them in the cookie-eating fun. But the doorbell continues to ring, bringing more and more children who are hungry for cookies. Soon, there are no more cookies to share, and the doorbell rings. What will the children do?

Estimation

Munsch, Robert. *Moira's Birthday* Moira wants to have a birthday party and invite all of the children in the school. And, without her parents' knowledge or consent, she does. What follows provides readers with a humorous look at estimation and problem solving.

Geometry Books

Tompert, Ann. *Grandfather Tang's Story*. (Suitable for Grades 3-6) Little Soo and her grandfather sat under the peach tree in their backyard making different shapes with tangram puzzles. She asks him to tell her a story about the fox fairies, Chou and Wu Ling, who were able to change their shapes. Grandfather Tang used the tangrams to tell his story, arranging the tans to show the different animals the fox fairies become.

Measurement Books
Lionni, Leo. *Inch by Inch*. A quick thinking inch worm saves his life by offering to measure the birds who want to eat him. Inch by inch, he measures the robin's tail, the flamingo's neck, the toucan's beak, the heron's legs, the pheasant's tail, and the hummingbird's body. But when he agrees to measure the nightingale's song, he takes the opportunity to inch away to freedom!

**Money Books**

Schwartz, David M. *If You Made a Million*. What does it mean to be a millionaire? Explore the world of money in this delightful book.

Silverstein, Shel. "The Googies Are Coming" from the book *Where the Sidewalk Ends*. Read the poem and arrange the child prices in order of most expensive to least expensive. Determine the cost of a bunch of children if purchased according to the Googies' price list.

Silverstein, Shel. *Where the Sidewalk Ends*. Determine exactly how much money the narrator had left by the end of the poem “Smart.”

Viorst, Judith. *Alexander, Who Used to Be Rich Last Sunday*. His grandparents gave him one dollar when they came to visit, and now he has nothing to show for it but a deck of cards with two cards missing, a one-eyed bear, a melted candle, and bus tokens. Also may be used when teaching subtraction.

**Multiplication Books**

Hong, Lily Toy. *Two of Everything*. (Grades 2-5). When an elderly couple digs up a large, old brass pot in their field, they discover it is magical. The pot doubles whatever the couple put in it! One day, Mrs. Hartak accidentally falls in the pot; the couple learns not everything in life should be doubled.

Mathews, Louise. *Bunches and Bunches of Bunnies*. Bunnies multiply right before children's eyes as they read this colorful rhyming book. One hundred and forty-four charming bunnies entertain readers as they teach basic multiplication facts!

Trivas, Irene. *Emma's Christmas*. In this lovely book, a prince gives the young and beautiful Emm all the gifts that are mentioned in the song, "The Twelve Days of Christmas" as many times as they are mentioned. It is a witty, delightful book that will provide children with many things to multiply.

**Place Value**

Fisher, Leonard Everett. *Number Art: Thirteen 123s From Around the World*. The author traces the development of various peoples and their number systems. He tells which systems are still in use, and calls attention to those which are more advanced—cultures using place value are considered more advanced. The number systems are illustrated making it easy for children to see how one system was derived from another.


**Problem Solving/Word Problems**

**WORD BANK**

*Try to use these math words in small-group discussions!*

<table>
<thead>
<tr>
<th>English</th>
<th>Spanish</th>
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<tr>
<td>coordinate</td>
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<td>inequality</td>
<td>desigualdad</td>
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<td>plot</td>
<td>trazar</td>
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<tr>
<td>variable</td>
<td>la variable</td>
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</table>
Math Discussion Sentence Starters

I think the best strategy to solve this is…
I would not solve with that strategy because…
I agree because…
I disagree because…
This problem is just like…
I think that strategy will work because…
I don’t think that strategy will work because…
If I try ___, then I think ___ will happen because…
I solved a problem like this before by…
Let me show you what I am thinking with a picture…
Let me show you what I am thinking with a chart…
Let me show you what I am thinking with these objects…
COMPLEX LANGUAGE AND LITERACY IN MATHEMATICS

CCSS requires teachers to infuse the four domains of language (i.e., listening, speaking, reading, writing) into mathematics instruction, which differs in complexity and demand from language and literacy in other content areas.
(A) Word/Phrase Level Language Demands

Various linguistic demands exist at the word- and phrase-level in mathematics for ELLs, such as synonyms, idioms, and multiple meaning words. Consider the following in your curriculum and in your own language use: (A) *synonyms* used to refer to the same math function (e.g., subtract, take away, minus, less), (B) *idioms* used in math problems (e.g., in the ballpark), and (C) *multiple meaning words* used in math contexts (e.g., table, round, root, mean, power). How can you support students with word- and phrase-level demands in math instruction?
(B) Sentence Level Language Demands

Various linguistic demands exist at the sentence level in mathematics for ELLs, such as language patterns and grammatical structures. Consider the following: (A) logical connectors (e.g., consequently, however) that in regular usage signal a logical relationship between parts of a text, but in math signal similarity or contradiction; (B) comparative structures (e.g., greater than, n times as much as); and (C) prepositions (e.g., divided by, divided into). How can you support students with sentence-level language demands in math instruction?
(C) Discourse Level Language Demands

Linguistic complexity exists at the discourse level in math for ELLs. (1) Semantic aspects pose difficulties, such as: *Three times a number is 2 more than 2 times the number. Find the numbers.* Solving this problem requires recognition of how many numbers are involved, the relationships between them, and which ones need to be identified. (2) ELLs encounter difficulties when they attempt to read and write mathematical sentences in the same way they read and write narrative text. How can you support students with discourse demands in math?
The Features of Academic Language in WIDA’s Standards

The Features of Academic Language operate within sociocultural contexts for language use.

<table>
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<th>Performance Criteria</th>
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<td>Linguistic Complexity</td>
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<td>Structure of speech/written text</td>
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<td>oral and written text)</td>
<td>Density of speech/written text</td>
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<td></td>
<td>Organization and cohesion of ideas</td>
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<td>Variety of sentence types</td>
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<tr>
<td>Sentence Level</td>
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<tr>
<td>Language Forms and Conventions</td>
<td>Types and variety of grammatical structures</td>
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<tr>
<td>(Types, array, and use of</td>
<td>Conventions, mechanics, and fluency</td>
</tr>
<tr>
<td>language structures)</td>
<td>Match of language forms to purpose/perspective</td>
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<tr>
<td>Word/Phrase Level</td>
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<tr>
<td>Vocabulary Usage</td>
<td>General, specific, and technical language</td>
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<td>(Specificity of word or</td>
<td>Multiple meanings of words and phrases</td>
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<td>Formulaic and idiomatic expressions</td>
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<td></td>
<td>Nuances and shades of meaning</td>
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<td></td>
<td>Collocations</td>
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</tbody>
</table>

The sociocultural contexts for language use involve the interaction between the student and the language environment, encompassing the…

- Register
- Genre/Text type
- Topic
- Task/Situation
- Participants’ identities and social roles
Lesson: Multiple-Meaning Words

1 Teach

Prepare visuals 8A and 8B to display. Copy visuals 8B and 8C for each student, Project db/pp/rugs.

Help students distinguish between words that are spelled alike but have different meanings.

- On the board or on chart paper, write the following sentence: The first batter in the game hit a home run. Ask students to read the sentence and tell what the word batter means. (a baseball player who hits the ball) Then write I like to taste the cake batter. Follow the same procedure. (mixture of ingredients)

- Ask: How do you know which meaning of the word batter is being used? Help students notice that context clues help them figure it out. If necessary, remind students that context clues are the words and sentences around a word that give hints about its meaning.

- Explain that students should consult a dictionary if context clues don’t help them with the meaning.
- Display visual 8A. Help students focus their attention on context clues to figure out the meanings of the underlined words. Have volunteers read each sentence aloud, and ask students to note that pronunciations of the underlined words are sometimes different.
- Point out to students that some multiple-meaning words have no connection to each other, such as desert (a sandy geographic area) and desert (to abandon). These words usually have separate entries in a dictionary. However, other multiple-meaning words are connected in some way. For example, words such as present or object are used both as nouns and as verbs. Their definitions are usually listed in the same dictionary entry because they are often just different ways to use the same word.
- Display visual 8B and give each student a copy. Students can use this chart to list multiple-meaning words, their meanings, and sample sentences. Get students started by sharing with them several of the words listed in Resources: Teacher Support for lesson: multiple-meaning words.

2 Practice

- Give each student a copy of visual 8C. Have them read the passage. Ask students to use context clues to figure out and write the meaning of each underlined word.
- Students may use a dictionary if they need help with this exercise.

3 Quick Assess

- Write sentences that contain multiple-meaning words (see Resources: Teacher Support for lesson: multiple-meaning words) and have students tell what each word means.
Differentiated Instruction

Extra Support
Reteaching Students may need more practice recognizing multiple-meaning words and using context clues to figure out their meanings. Write the following sentences on the board or on chart paper. Have students use context clues to find the meaning of each underlined word, and then verify the meaning by looking up the word in a dictionary. Ask them to write the meaning of the word as it is used in each sentence.

Both yards have a swing set. The box is three yards long. The office building is four stories tall. The old shirt had many rips and tears in it. Wait a second before you go. My painting came in second at the art fair. I didn’t mean to hurt his feelings. Be careful not to say mean things to someone in anger.

Language Support
How Do You Say It? Students may need practice with multiple-meaning words that are pronounced differently. Remind students that even words that are spelled the same way may be pronounced differently. Write the following sentences on the board or on chart paper, read them aloud, and have students read them with you. Then work with them to use context clues and a dictionary to figure out the meaning of each underlined word. Point out that the meaning of the word in context tells them how to pronounce the word. Please close the door. My grandmother likes it when I sit close to her. The content of that TV program is always silly. I feel at peace and content with the work today. The Arizona desert is a fascinating place. Animals in the wild rarely desert their young.

Provide students with additional copies of visual 8B on which they can list confusing multiple-meaning words, their meanings, and sample sentences to use as a reference. Or they may want to put the words on index cards and add sketches to help them remember the word meanings. Encourage students to begin their lists with the words in this lesson.

Extra Challenge
Do Some Research Explain to students that there are thousands of English words that have multiple meanings. Challenge them to use a dictionary or the Internet to find at least five such words not yet discussed in class. Ask them to write the words they find, at least two different definitions for each one, and an example sentence for each definition. They may want to use additional copies of visual 8B to record the information. Ask them to share their results with the rest of the class, perhaps by posting them on the bulletin board so that others can add them to their own reference lists.

Resources

Teacher Support
Definitions Technically, multiple-meaning words and homographs are not the same. However, the differences between them are usually taught at higher grade levels. It is not necessary to have students differentiate between them.

A multiple-meaning word has several meanings that are listed under one entry in a dictionary. Often the same word is used as more than one part of speech, such as a noun and a verb. All uses of the word have the same word history, or etymology, even though the pronunciations may differ. Here are some multiple-meaning words that fifth-grade students may encounter:

- break (transitive verb)
- corner (noun, transitive verb)
- crash (transitive verb)
- crash (noun)
- dare (transitive verb)
- dare (noun)
- fire (noun, transitive verb)
- guard (noun, transitive verb)
- notice (noun, transitive verb)
- pack (noun, transitive verb)
- pack (transitive verb)
- raise (transitive verb)
- shape (noun, transitive verb)
- shape (verb)
- signal (noun, transitive verb)
- thunder (noun, transitive verb)

Homographs are words that are spelled alike but listed as separate entries in a dictionary. They may or may not be pronounced the same way. However, homographs always have different meanings and different etymologies. Here are some homographs that fifth-grade students may encounter:

- affect (verb)
- effect (noun, verb)
- arms (noun, plural)
- arms (verb)
- bluff (noun)
- bluff (verbal)
- converse (verb)
- converse (noun)

Double Trouble Ask students to refer to their lists of multiple-meaning words. Challenge them to develop a single sentence that includes two different meanings of the same word. To get students started, give them an example, such as The fly on the wall will fly away. Invite them to write as many sentences as possible. Have students share their sentences by reading them aloud or displaying them on a bulletin board. Have other students try to use context clues to determine the word’s meanings.

Making Riddles Encourage pairs of students to work together to write riddles that include multiple meanings of a word. Challenge them to develop a riddle that includes multiple meanings of a word that they want an audience to guess. An example is I quack, and I lower my head quickly! What am I? (duck) Have pairs read their riddles aloud or act them out as the rest of the class tries to solve each one.

Add It to the List! Display on the bulletin board a large piece of poster paper with the heading Multiple-Meaning Words, under which students may write examples of multiple-meaning words that they come across in their reading or that they hear on the television or radio. Students should write a meaning for each use of the words. Let students use the list as a class reference.

Additional Activities

Cross-References

Context Clues, folder 6
Homophones, folder 7
**Differentiated Instruction**

**Extra Support**

Reteaching: Students may need more practice recognizing multiple-meaning words and using context clues to figure out their meanings. Write the following sentences on the board or on chart paper. Have students use context clues to find the meaning of each underlined word, and then verify the meaning by looking up the word in a dictionary. Ask them to write the meaning of the word as it is used in each sentence.

- *Both yards have a swing set.*
- *The old shirt had many rips and tears.*
- *The sad movie had us in tears.*
- *I read stories to my little brother.*
- *The office building is four stories tall.*
- *I didn’t mean to hurt his feelings.*
- *My painting came in second at the art fair.*
- *The Arizona desert is a fascinating place.*
- *7/10/06 10:13:32 AM*

**Language Support**

How Do You Say It? Students may need practice with multiple-meaning words that are pronounced differently. Remind students that even words that are spelled the same way may be pronounced differently. Write the following sentences on the board or on chart paper, read them aloud, and have students read them with you. Then work with them to use context clues and a dictionary to figure out the meaning of each underlined word. Point out that the meaning of the word in context tells them how to pronounce the word.

- *Please close the door.*
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- *I feel at peace and content with the work today.*
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Provide students with additional copies of visual 8B on which they can list confusing multiple-meaning words, their meanings, and sample sentences to use as a reference. Or they may want to put the words on index cards and add sketches to help them remember the word meanings. Encourage students to begin their lists with the words in this lesson.

**Extra Challenge**

Do Some Research Explain to students that there are thousands of English words that have multiple meanings. Challenge them to use a dictionary or the Internet to find at least five such words not yet discussed in class. Ask them to write the words they find, at least two different definitions for each one, and an example sentence for each definition. They may want to use additional copies of visual 8B to record the information. Ask them to share their results with the rest of the class, perhaps by posting them on the bulletin board so that others can add them to their own reference lists.

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**Resources**

**Teacher Support**

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- notice
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- shape
- signal
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- bluff
- converse
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- fine
- hamper
- hour
- minute
- palm
- pole

- influence
- parts
- fool
- opposite
- delight
- good
- hold
- very
- kind
- end
- piece

**Additional Activities**

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**Add It to the List!**

Display on the bulletin board a large piece of poster paper with the heading Multiple-Meaning Words, under which students may write examples of multiple-meaning words that they come across in their reading or that they hear on the television or radio. Students should write a meaning for each use of the words. Let students use the list as a class reference.

**Cross-References**

- Content Class, folder 6
- Homophones, folder 7

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© Great Source. All rights reserved. (5) Lessons in Literacy
Each visual included in this folder is shown here for quick and easy reference. In addition to the suggestions in the lesson, the visuals are designed to be versatile, allowing you to customize their use. For your convenience, the visuals are available in three formats: transparencies, designed to be versatile, allowing you to customize their use. For your reference. In addition to the suggestions in the lesson, the visuals are each included in this folder is shown here for quick and easy reference. In addition to the suggestions in the lesson, the visuals are each included in this folder is shown here for quick and easy reference. In addition to the suggestions in the lesson, the visuals are each included in this folder is shown here for quick and easy reference. In addition to the suggestions in the lesson, the visuals are each included in this folder is shown here for quick and easy reference. In addition to the suggestions in the lesson, the visuals are each included in this folder is shown here for quick and easy reference. In addition to the suggestions in the lesson, the visuals are
MULTIPLE REPRESENTATIONS

Math CCSS relies on resources and realia, such as charts, visuals, drawings, objects, and gestures, for teachers to support conceptual development and for students to share multiple representations of the same problem.
(A) Sequenced Pictures

ELLs can show understanding by drawing or arranging a set of pictures. Sequenced pictures can depict steps or stages, or they can be as simple as before and after. After the pictures have been drawn or arranged, students can supplement the pictures with additional information. Beginners can label parts of each picture, with or without a word bank. Intermediate ELLs can add descriptive and explanatory words, phrases, and sentences. Look at the examples to consider and discuss: (1) How do pictures demonstrate students’ content mastery? (2) How do pictures support language development?
(B) Graphic Organizers

Whereas graphic organizers are no doubt a part of your regular teaching repertoire, consider use for students to represent understanding, such as techniques to cluster, problem solve, compare, and sequence. As alternative assignments, they allow ELLs to convey a large amount of content information in a linguistically simplified form. Look at the examples to consider and discuss: (1) How do graphic organizers demonstrate students’ content mastery? (2) How do graphic organizers support students’ language development?
(C) Total Physical Response

Using gestures and movement in content instruction are effective for ELLs, particularly beginners, both pre- and post-CCSS. Total physical response (TPR) engages teachers and learners in authentic language development through the focus on the total response of language users (e.g., gestures, facial expressions, movements). Please: (1) Read the steps to TPR use with ELLs, and (2) Watch the video of TPR in an algebra lesson. Discuss: How could you use TPR in your math instruction to allow students to portray their understanding with more than static language?
Parts of a wave

wavelength

trough

crest

Speed

Big waves move fast

Little waves move slowly

Temperature

Waves mix deep water (cold) with surface water (warm).

Figure 10.10  All About Waves
**Figure 10.12** The Cluster or Web

**Figure 10.13** The Web in Action: Nomads
Figure 10.14  The Problem-Solving Organizer

Figure 10.15  The Problem-Solving Organizer in Action: Water Pollution in the United States
Figure 10.17 Venn Diagram Comparing Two Things

Figure 10.18 Venn Diagram Comparing Three Things
<table>
<thead>
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<th>Qualities →</th>
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<th>1</th>
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</table>

*Figure 10.20  The Matrix*
<table>
<thead>
<tr>
<th>Specimen</th>
<th>Luster</th>
<th>Cleavage</th>
<th>Hardness</th>
<th>Color</th>
<th>Other</th>
</tr>
</thead>
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<td>E</td>
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</tbody>
</table>

Figure 10.21 The Matrix in Science: Comparing Minerals
I. Introduction

It is a remarkable phenomenon that children can learn to speak without ever being consciously aware of the sophisticated grammar they are using. Indeed, adults too can live a perfectly satisfactory life without ever thinking about ideas such as parts of speech, subjects, predicates, or subordinate clauses. Both children and adults can easily recognize ungrammatical sentences, at least if the mistake is not too subtle, and to do this it is not necessary to be able to explain the rules that have been violated. Nevertheless, there is no doubt that one’s understanding of language is hugely enhanced by a knowledge of basic grammar, and this understanding is essential for anybody who wants to do more with language than use it unreflectingly as a means to a nonlinguistic end.

The same is true of mathematical language. Up to a point, one can do and speak mathematics without knowing how to classify the different sorts of words one is using, but many of the sentences of advanced mathematics have a complicated structure that is much easier to understand if one knows a few basic terms of mathematical grammar. The object of this section is to explain the most important mathematical “parts of speech,” some of which are similar to those of natural languages and others quite different. These are normally taught right at the beginning of a university course in mathematics. Much of The Companion can be understood without a precise knowledge of mathematical grammar, but a careful reading of this article will help the reader who wishes to follow some of the later, more advanced parts of the book.

The main reason for using mathematical grammar is that the statements of mathematics are supposed to be completely precise, and it is not possible to achieve complete precision unless the language one uses is free of many of the vaguenesses and ambiguities of ordinary speech. Mathematical sentences can also be highly complex: if the parts that made them up were not clear and simple, then the unclarities would rapidly accumulate and render the sentences unintelligible.

To illustrate the sort of clarity and simplicity that is needed in mathematical discourse, let us consider the famous mathematical sentence “Two plus two equals four” as a sentence of English rather than of mathematics, and try to analyze it grammatically. On the face of it, it contains three nouns (“two,” “two,” and “four”), a verb (“equals”) and a conjunction (“plus”). However, looking more carefully we may begin to notice some oddities. For example, although the word “plus” resembles the word “and,” the most obvious example of a conjunction, it does not behave in quite the same way, as is shown by the sentence “Mary and Peter love Paris.” The verb in this sentence, “love,” is plural, whereas the verb in the previous sentence, “equals,” was singular. So the word “plus” seems to take two objects (which happen to be numbers) and produce out of them a new, single object, while “and” conjoins “Mary” and “Peter” in a looser way, leaving them as distinct people.

Reflecting on the word “and” a bit more, one finds that it has two very different uses. One, as above, is to link two nouns, whereas the other is to join two whole sentences together, as in “Mary likes Paris and Peter likes New York.” If we want the basics of our language to be absolutely clear, then it will be important to be aware of this distinction. (When mathematicians are at their most formal, they simply outlaw the noun-linking use of “and”—a sentence such as “3 and 5 are prime numbers” is then paraphrased as “3 is a prime number and 5 is a prime number.”)

This is but one of many similar questions: anybody who has tried to classify all words into the standard eight parts of speech will know that the classification is hopelessly inadequate. What, for example, is the role of the word “six” in the sentence “This section has six subsections”? Unlike “two” and “four” earlier, it is certainly not a noun. Since it modifies the noun “subsection” it would traditionally be classified as an adjective, but it does not behave like most adjectives: the sentences “My car is not very fast” and “Look at that tall building” are perfectly grammatical, whereas the sentences “My car is not very six” and “Look at that six building” are not just nonsense but ungrammatical nonsense. So do we classify adjectives further into numerical adjectives and nonnumerical adjectives? Perhaps we do, but then our troubles will be only just beginning. For example, what about possessive adjectives such as “my” and “your”? In general, the more one tries to refine the classification of English words, the more one realizes how many different grammatical roles there are.
2 Four Basic Concepts

Another word that famously has three quite distinct meanings is "is." The three meanings are illustrated in the following three sentences.

(1) 5 is the square root of 25.
(2) 5 is less than 10.
(3) 5 is a prime number.

In the first of these sentences, "is" could be replaced by "equals": it says that two objects, 5 and the square root of 25, are in fact one and the same object, just as it does in the English sentence "London is the capital of the United Kingdom." In the second sentence, "is" plays a completely different role. The words "less than 10" form an adjectival phrase, specifying a property that numbers may or may not have, and "is" in this sentence is like "is" in the English sentence "Grass is green." As for the third sentence, the word "is" there means "is an example of," as it does in the English sentence "Mercury is a planet."

These differences are reflected in the fact that the sentences cease to resemble each other when they are written in a more symbolic way. An obvious way to write sentence (1) is $5 = \sqrt{25}$. As for (2), it would usually be written $5 < 10$, where the symbol "<" means "is less than." The third sentence would normally not be written symbolically because the concept of a prime number is not quite basic enough to have universally recognized symbols associated with it. However, it is sometimes useful to do so, and then one must invent a suitable symbol. One way to do it would be to adopt the convention that if $n$ is a positive integer, then $P(n)$ stands for the sentence "$n$ is prime." Another way, which does not hide the word "is," is to use the language of sets.

2.1 Sets

Broadly speaking, a set is a collection of objects, and in mathematical discourse these objects are mathematical ones such as numbers, points in space, or even other sets. If we wish to rewrite sentence (3) symbolically, another way to do it is to define $P$ to be the collection, or set, of all prime numbers. Then we can rewrite it as "5 belongs to the set $P." This notion of belonging to a set is sufficiently basic to deserve its own symbol, and the symbol used is "\in." So a fully symbolic way of writing the sentence is $5 \in P$.

The members of a set are usually called its elements, and the symbol "\in" is usually read "is an element of." So the "is" of sentence (3) is more like "\in" than "=.

Although one cannot directly substitute the phrase "is an element of" for "is," one can do so if one is prepared to modify the rest of the sentence a little.

There are three common ways to denote a specific set. One is to list its elements inside curly brackets: \{2, 3, 5, 7, 11, 13, 17, 19\}, for example, is the set whose elements are the eight numbers 2, 3, 5, 7, 11, 13, 17, and 19. The majority of sets considered by mathematicians are too large for this to be feasible—indeed, they are often infinite—so a second way to denote sets is to use dots to imply a list that is too long to write down: for example, the expressions \{1, 2, 3, ..., 10\} and \{2, 4, 6, 8, ...\} can be used to represent the set of all positive integers up to 100 and the set of all positive even numbers, respectively. A third way, and the way that is most important, is to define a set via a property: an example that shows how this is done is the expression \{x : x is prime and x < 20\}. To read an expression such as this, one first reads the opening curly bracket as "The set of." Next, one reads the symbol that occurs before the colon. The colon itself one reads as "such that." Finally, one reads what comes after the colon, which is the property that determines the elements of the set. In this instance, we end up saying, "The set of $x$ such that $x$ is prime and $x$ is less than 20," which is in fact equal to the set \{2, 3, 5, 7, 11, 13, 17, 19\} considered earlier.

Many sentences of mathematics can be rewritten in set-theoretic terms. For example, sentence (2) earlier could be written as $5 \in \{n : n < 10\}$. Often there is no point in doing this (as here, where it is much easier to write 5 < 10) but there are circumstances where it becomes extremely convenient. For example, one of the great advances in mathematics was the use of Cartesian coordinates to translate geometry into algebra and the way this was done was to define geometrical objects as sets of points, where points were themselves defined as pairs or triples of numbers. So, for example, the set \{(x, y) : x^2 + y^2 = 1\} is (or represents) a circle of radius 1 with its center at the origin (0, 0). That is because, by the Pythagorean theorem, the distance from (0, 0) to (x, y) is $\sqrt{x^2 + y^2}$, so the sentence "$x^2 + y^2 = 1$" can be reexpressed geometrically as "the distance from (0, 0) to (x, y) is 1." If all we ever cared about was which points were in the circle, then we could make do with sentences such as "$x^2 + y^2 = 1$," but in geometry one often wants to consider the entire circle as a single object (rather than as a multiplicity of points, or as a property that points might have), and then set-theoretic language is indispensable.
A second circumstance where it is usually hard to do
without sets is when one is defining new mathematical
objects. Very often such an object is a set together with
a mathematical structure imposed on it, which takes the
form of certain relationships among the elements of the
set. For examples of this use of set-theoretic language,
see sections 1 and 2, on number systems and algebraic
structures, respectively, in SOME FUNDAMENTAL
MATHEMATICAL DEFINITIONS [I.3].

Sets are also very useful if one is trying to do meta-
mathematics, that is, to prove statements not about
mathematical objects but about the process of math-
ematical reasoning itself. For this it helps a lot if one
can devise a very simple language—with a small vocab-
ulary and an uncomplicated grammar—into which it is
in principle possible to translate all mathematical
arguments. Sets allow one to reduce greatly the num-
ber of parts of speech that one needs, turning almost
all of them into nouns. For example, with the help
of the membership symbol \( \in \) one can do without
adjectives, as the translation of "5 is a prime number"
(where "prime" functions as an adjective) into "5 \( \in P \)
has already suggested.\(^1\) This is of course an artificial
process—imagine replacing "roses are red" by "roses
belong to the set \( R \)"—but in this context it is not impor-
tant for the formal language to be natural and easy to
understand.

2.2 Functions

Let us now switch attention from the word "is" to some
other parts of the sentences (1)–(3), focusing first on
the phrase “the square root of” in sentence (1). If we
wish to think about this phrase grammatically, then we
should analyze what sort of role it plays in a sentence,
and the analysis is simple: in virtually any mathematical
sentence where the phrase appears, it is followed by
the name of a number. If the number is \( n \), then this
produces the slightly longer phrase, “the square root
of \( n \),” which is a noun phrase that denotes a number
and plays the same grammatical role as a number (at
least when the number is used as a noun rather than as
an adjective). For instance, replacing “5” by “the square
root of 25” in the sentence “5 is less than 7” yields a
new sentence, “The square root of 25 is less than 7,”
that is still grammatically correct (and true).

One of the most basic activities of mathematics is
to take a mathematical object and transform it into
another one, sometimes of the same kind and some-
times not. “The square root of” transforms numbers
into numbers, as do “four plus,” “two times,” “the
cosine of,” and “the logarithm of.” A nonnumerical
example is “the center of gravity of,” which transforms
geometrical shapes (provided they are not too exotic or
complicated to have a center of gravity) into points—
meaning that if \( S \) stands for a shape, then “the center
of gravity of \( S \)” stands for a point. A function is, roughly
speaking, a mathematical transformation of this kind.

It is not easy to make this definition more precise. To
ask, “What is a function?” is to suggest that the answer
should be a thing of some sort, but functions seem to
be more like processes. Moreover, when they appear in
mathematical sentences they do not behave like nouns.
(They are more like prepositions, though with a definite
difference that will be discussed in the next subsec-
ction.) One might therefore think it inappropriate to ask
what kind of object “the square root of” is. Should one
not simply be satisfied with the grammatical analysis
already given?

As it happens, no. Over and over again, through-
out mathematics, it is useful to think of a mathemati-
cal phenomenon, which may be complex and very un-
thinglike, as a single object. We have already seen a sim-
ple example: a collection of infinitely many points in the
plane or space is sometimes better thought of as a sin-
gle geometrical shape. Why should one wish to do this
for functions? Here are two reasons. First, it is conve-
nient to be able to say something like, “The derivative
of \( \sin \) is \( \cos \),” or to speak in general terms about some
functions being differentiable and others not. More gen-
erally, functions can have properties, and in order to
discuss those properties one needs to think of func-
tions as things. Second, many algebraic structures are
most naturally thought of as sets of functions. (See,
for example, the discussion of groups and symmetry
in [I.3 §2.1]. See also HILBERT SPACES [III.37], FUNCTION
SPACES [III.29], and VECTOR SPACES [I.3 §2.3].)

If \( f \) is a function, then the notation \( f(x) = y \) means
that \( f \) turns the object \( x \) into the object \( y \). Once one
starts to speak formally about functions, it becomes
important to specify exactly which objects are to be
subjected to the transformation in question, and what
sort of objects they can be transformed into. One of
the main reasons for this is that it makes it possible to
discuss another notion that is central to mathematics,
that of inverting a function. (See [I.4 §1] for a discussion
of why it is central.) Roughly speaking, the inverse of a
function is another function that undoes it, and that it

\(^1\) For another discussion of adjectives see ARITHMETIC GEOMETRY
[IV.5 §3.1].
undoes; for example, the function that takes a number \( n \) to \( n - 4 \) is the inverse of the function that takes \( n \) to \( n + 4 \), since if you add four and then subtract four, or vice versa, you get the number you started with.

Here is a function \( f \) that cannot be inverted. It takes each number and replaces it by the nearest multiple of 100, rounding up if the number ends in 50. Thus, \( f(113) = 100 \), \( f(3879) = 3900 \), and \( f(1050) = 1100 \). It is clear that there is no way of undoing this process with a function \( g \). For example, in order to undo the effect of \( f \) on the number 113 we would need \( g(100) \) to equal 113. But the same argument applies to every number that is at least as big as 50 and smaller than 150, and \( g(100) \) cannot be more than one number at once.

Now let us consider the function that doubles a number. Can this be inverted? Yes it can, one might say: just divide the number by two again. And much of the time this would be a perfectly sensible response, but not, for example, if it was clear from the context that the numbers being talked about were positive integers. Then one might be focusing on the difference between even and odd numbers, and this difference could be encapsulated by saying that odd numbers are precisely those numbers \( n \) for which the equation \( 2x = n \) does not have a solution. (Notice that one can undo the doubling process by halving. The problem here is that the relationship is not symmetrical: there is no function that can be undone by doubling, since you could never get back to an odd number.)

To specify a function, therefore, one must be careful to specify two sets as well: the domain, which is the set of objects to be transformed, and the range, which is the set of objects they are allowed to be transformed into. A function \( f \) from a set \( A \) to a set \( B \) is a rule that specifies, for each element \( x \) of \( A \), an element \( y = f(x) \) of \( B \). Not every element of the range needs to be used: consider once again the example of “two times” when the domain and range are both the set of all positive integers. The set \( \{ f(x) : x \in A \} \) of values actually taken by \( f \) is called the image of \( f \). (Slightly confusingly, the word “image” is also used in a different sense, applied to the individual elements of \( A \); if \( x \in A \), then its image is \( f(x) \).)

The following symbolic notation is used. The expression \( f : A \rightarrow B \) means that \( f \) is a function with domain \( A \) and range \( B \). If we then write \( f(x) = y \), we know that \( x \) must be an element of \( A \) and \( y \) must be an element of \( B \). Another way of writing \( f(x) = y \) that is sometimes more convenient is \( f : x \mapsto y \). (The bar on the arrow is to distinguish it from the arrow in \( f : A \rightarrow B \), which has a very different meaning.)

If we want to undo the effect of a function \( f : A \rightarrow B \), then we can, as long as we avoid the problem that occurred with the approximating function discussed earlier. That is, we can do it if \( f(x) \) and \( f(x') \) are different whenever \( x \) and \( x' \) are different elements of \( A \). If this condition holds, then \( f \) is called an injection. On the other hand, if we want to find a function \( g \) that is undone by \( f \), then we can do so as long as we avoid the problem of the integer-doubling function. That is, we can do it if every element \( y \) of \( B \) is equal to \( f(x) \) for some element \( x \) of \( A \) (so that we have the option of setting \( g(y) = x \)). If this condition holds, then \( f \) is called a surjection. If \( f \) is both an injection and a surjection, then \( f \) is called a bijection. Bijections are precisely the functions that have inverses.

It is important to realize that not all functions have tidy definitions. Here, for example, is the specification of a function from the positive integers to the positive integers: \( f(n) = n \) if \( n \) is a prime number, \( f(n) = k \) if \( n \) is of the form \( 2k \) for an integer \( k \) greater than 1, and \( f(n) = 13 \) for all other positive integers \( n \). This function has an unpleasant, arbitrary definition but it is nevertheless a perfectly legitimate function. Indeed, “most” functions, though not most functions that one actually uses, are so arbitrary that they cannot be defined. (Such functions may not be useful as individual objects, but they are needed so that the set of all functions from one set to another has an interesting mathematical structure.)

2.3 Relations

Let us now think about the grammar of the phrase “less than” in sentence (2). As with “the square root of,” it must always be followed by a mathematical object (in this case a number again). Once we have done this we obtain a phrase such as “less than \( n \),” which is importantly different from the “square root of \( n \)” because it behaves like an adjective rather than a noun, and refers to a property rather than an object. This is just how prepositions behave in English: look, for example, at the word “under” in the sentence “The cat is under the table.”

At a slightly higher level of formality, mathematicians like to avoid too many parts of speech, as we have already seen for adjectives. So there is no symbol for “less than”; instead, it is combined with the previous word “is” to make the phrase “is less than,” which is
denoted by the symbol "<." The grammatical rules for this symbol are once again simple. To use "<" in a sentence, one should precede it by a noun and follow it by a noun. For the resulting grammatically correct sentence to make sense, the nouns should refer to numbers (or perhaps to more general objects that can be put in order). A mathematical "object" that behaves like this is called a relation, though it might be more accurate to call it a potential relationship. "Equals" and "is an element of" are two other examples of relations.

As with functions, it is important, when specifying a relation, to be careful about which objects are to be related. Usually a relation comes with a set $A$ of objects that may or may not be related to each other. For example, the relation "<" might be defined on the set of all positive integers, or alternatively on the set of all real numbers; strictly speaking these are different relations. Sometimes relations are defined with reference to two sets $A$ and $B$. For example, if the relation is "$\in$," then $A$ might be the set of all positive integers and $B$ the set of all sets of positive integers.

There are many situations in mathematics where one wishes to regard different objects as "essentially the same," and to help us make this idea precise there is a very important class of relations known as equivalence relations. Here are two examples. First, in elementary geometry one sometimes cares about shapes but not about sizes. Two shapes are said to be similar if one can be transformed into the other by a combination of reflections, rotations, translations, and enlargements (see figure 1); the relation "is similar to" is an equivalence relation. Second, when doing arithmetic modulo $m$ [III.59], one does not wish to distinguish between two whole numbers that differ by a multiple of $m$: in this case one says that the numbers are congruent (mod $m$); the relation "is congruent (mod $m$) to" is another equivalence relation.

What exactly is it that these two relations have in common? The answer is that they both take a set (in the first case the set of all geometrical shapes, and in the second the set of all whole numbers) and split it into parts, called equivalence classes, where each part consists of objects that one wishes to regard as essentially the same. In the first example, a typical equivalence class is the set of all shapes that are similar to some given shape; in the second, it is the set of all integers that leave a given remainder when you divide by $m$ (for example, if $m = 7$ then one of the equivalence classes is the set $\{\ldots, -16, -9, -2, 5, 12, 19, \ldots\}$).

An alternative definition of what it means for a relation $\sim$, defined on a set $A$, to be an equivalence relation is that it has the following three properties. First, it is reflexive, which means that $x \sim x$ for every $x$ in $A$. Second, it is symmetric, which means that if $x$ and $y$ are elements of $A$ and $x \sim y$, then it must also be the case that $y \sim x$. Third, it is transitive, meaning that if $x$, $y$, and $z$ are elements of $A$ such that $x \sim y$ and $y \sim z$, then it must be the case that $x \sim z$. (To get a feel for these properties, it may help if you satisfy yourself that the relations "is similar to" and "is congruent (mod $m$) to" both have all three properties, while the relation "<," defined on the positive integers, is transitive but neither reflexive nor symmetric.)

One of the main uses of equivalence relations is to make precise the notion of \textsc{quotient} [I.3 §3.3] constructions.

2.4 Binary Operations

Let us return to one of our earlier examples, the sentence "Two plus two equals four." We have analyzed the word "equals" as a relation, an expression that sits between the noun phrases "two plus two" and "four" and makes a sentence out of them. But what about "plus"? That also sits between two nouns. However, the result, "two plus two," is not a sentence but a noun phrase. That pattern is characteristic of \textit{binary operations}. Some familiar examples of binary operations are "plus," "minus," "times," "divided by," and "raised to the power."

As with functions, it is customary, and convenient, to be careful about the set to which a binary operation is applied. From a more formal point of view, a binary operation on a set $A$ is a function that takes pairs of elements of $A$ and produces further elements of $A$ from them. To be more formal still, it is a function with the set of all pairs $(x, y)$ of elements of $A$ as its domain.
and with $A$ as its range. This way of looking at it is not reflected in the notation, however, since the symbol for the operation comes between $x$ and $y$ rather than before them: we write $x + y$ rather than $+(x, y)$.

There are four properties that a binary operation may have that are very useful if one wants to manipulate sentences in which it appears. Let us use the symbol $\ast$ to denote an arbitrary binary operation on some set $A$. The operation $\ast$ is said to be commutative if $x \ast y$ is always equal to $y \ast x$, and associative if $x \ast (y \ast z)$ is always equal to $(x \ast y) \ast z$. For example, the operations “plus” and “times” are commutative and associative, whereas “minus,” “divided by,” and “raised to the power” are neither (for instance, $9 - (5 - 3) = 7$ while $(9 - 5) - 3 = 1$). These last two operations raise another issue: unless the set $A$ is chosen carefully, they may not always be defined. For example, if one restricts one’s attention to the positive integers, then the expression $3 - 5$ has no meaning. There are two conventions one could imagine adopting in response to this. One might decide not to insist that a binary operation should be defined for every pair of elements of $A$, and to regard it as a desirable extra property of an operation if it is defined everywhere. But the convention actually in force is that binary operations do have to be defined everywhere, so that “minus,” though a perfectly good binary operation on the set of all integers, is not a binary operation on the set of all positive integers.

An element $e$ of $A$ is called an identity for $\ast$ if $e \ast x = x \ast e = x$ for every element $x$ of $A$. The two most obvious examples are 0 and 1, which are identities for “plus” and “times,” respectively. Finally, if $\ast$ has an identity $e$ and $x$ belongs to $A$, then an inverse for $x$ is an element $y$ such that $x \ast y = y \ast x = e$. For example, if $\ast$ is “plus” then the inverse of $x$ is $-x$, while if $\ast$ is “times” then the inverse is $1/x$.

These basic properties of binary operations are fundamental to the structures of abstract algebra. See four important algebraic structures [I.3 §2] for further details.

3 Some Elementary Logic

3.1 Logical Connectives

A logical connective is the mathematical equivalent of a conjunction. That is, it is a word (or symbol) that joins two sentences to produce a new one. We have already discussed an example, namely “and” in its sentence-linking meaning, which is sometimes written by the symbol “$\land$,” particularly in more formal or abstract mathematical discourse. If $P$ and $Q$ are statements (note here the mathematical habit of representing not just numbers but any objects whatsoever by single letters), then $P \land Q$ is the statement that is true if and only if both $P$ and $Q$ are true.

Another connective is the word “or,” a word that has a more specific meaning for mathematicians than it has for normal speakers of the English language. The mathematical use is illustrated by the tiresome joke of responding, “Yes please,” to a question such as, “Would you like your coffee with or without sugar?” The symbol for “or,” if one wishes to use a symbol, is “$\lor$,” and the statement $P \lor Q$ is true if and only if $P$ is true or $Q$ is true. This is taken to include the case when they are both true, so “or,” for mathematicians, is always the so-called inclusive version of the word.

A third important connective is “implies,” which is usually written “$\Rightarrow$.” The statement $P \Rightarrow Q$ means, roughly speaking, that $Q$ is a consequence of $P$, and is sometimes read as “if $P$ then $Q$.” However, as with “or,” this does not mean quite what it would in English. To get a feel for the difference, consider the following even more extreme example of mathematical pedantry. At the supper table, my young daughter once said, “Put your hand up if you are a girl.” One of my sons, to tease her, put his hand up on the grounds that, since she had not added, “and keep it down if you are a boy,” his doing so was compatible with her command.

Something like this attitude is taken by mathematicians to the word “implies,” or to sentences containing the word “if.” The statement $P \Rightarrow Q$ is considered to be true under all circumstances except one: it is not true if $P$ is true and $Q$ is false. This is the definition of “implies.” It can be confusing because in English the word “implies” suggests some sort of connection between $P$ and $Q$, that $P$ in some way causes $Q$ or is at least relevant to it. If $P$ causes $Q$ then certainly $P$ cannot be true without $Q$ being true, but all a mathematician cares about is this logical consequence and not whether there is any reason for it. Thus, if you want to prove that $P \Rightarrow Q$, all you have to do is rule out the possibility that $P$ could be true and $Q$ false at the same time. To give an example: if $n$ is a positive integer, then the statement “$n$ is a perfect square with final digit 7” implies the statement “$n$ is a prime number,” not because there is any connection between the two but because no perfect square ends in a 7. Of course, implications of this kind are less interesting mathematically than more genuine-seeming ones, but the reward for accepting them is that, once again, one
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avoids being confused by some of the ambiguities and subtle nuances of ordinary language.

3.2 Quantifiers

Yet another ambiguity in the English language is exploited by the following old joke that suggests that our priorities need to be radically rethought.

(4) Nothing is better than lifelong happiness.
(5) But a cheese sandwich is better than nothing.
(6) Therefore, a cheese sandwich is better than lifelong happiness.

Let us try to be precise about how this play on words works (a good way to ruin any joke, but not a tragedy in this case). It hinges on the word “nothing,” which is used in two different ways. The first sentence means “There is no single thing that is better than lifelong happiness,” whereas the second means “It is better to have something that we like to drink, even if that something varies from person to person. The precise formulations that capture the difference are as follows.

(7) Everybody likes at least one drink, namely water.
(8) Everybody likes at least one drink; I myself go for red wine.

The first sentence makes the point (not necessarily correctly) that there is one drink that everybody likes, whereas the second claims merely that we all have something we like to drink, even if that something varies from person to person. The precise formulations that capture the difference are as follows.

(7′) There exists a drink D such that, for every person P, P likes D.
(8′) For every person P there exists a drink D such that P likes D.

This illustrates an important general principle: if you take a sentence that begins “for every x there exists y such that …” and interchange the two parts so that it now begins “there exists y such that, for every x, …,” then you obtain a much stronger statement, since y is no longer allowed to depend on x. If the second statement is still true—that is, if you really can choose a y that works for all the x at once—then the first statement is said to hold uniformly.

The symbols ∀ and ∃ are often used to stand for “for all” and “there exists,” respectively. This allows us to write quite complicated mathematical sentences in a highly symbolic form if we want to. For example, suppose we let P be the set of all primes, as we did earlier. Then the following symbols make the claim that there are infinitely many primes, or rather a slightly different claim that is equivalent to it.

(9) ∀n ∃m (m > n) ∧ (m ∈ P).

In words, this says that for every n we can find some m that is both bigger than n and a prime. If we wish to unpack sentence (6) further, we could replace the part m ∈ P by

(10) ∀a,b ab = m ⇒ ((a = 1) ∨ (b = 1)).

There is one final important remark to make about the quantifiers “∀” and “∃.” I have presented them as if they were freestanding, but actually a quantifier is always associated with a set (one says that it quantifies over that set). For example, sentence (10) would not be a translation of the sentence “m is prime” if a and b were allowed to be fractions: if a = 3 and b = 2/3 then ab = 7 without either a or b equaling 1, but this does not show that 7 is not a prime. Implicit in the opening symbols ∀a,b is the idea that a and b are intended to be positive integers. If this had not been clear from the context, then we could have used the symbol N (which stands for the set of all positive integers) and started sentence (10) with ∀a,b ∈ N instead.
3.3 Negation

The basic idea of negation in mathematics is very simple: there is a symbol, “¬,” which means “not,” and if \( P \) is any mathematical statement, then \( \neg P \) stands for the statement that is true if and only if \( P \) is not true. However, this is another example of a word that has a slightly more restricted meaning to mathematicians than it has in ordinary speech.

To illustrate this phenomenon once again, let us take \( A \) to be a set of positive integers and ask ourselves what the negation is of the sentence “Every number in the set \( A \) is odd.” Many people when asked this question will suggest, “Every number in the set \( A \) is even.” However, this is wrong: if one thinks carefully about what exactly would have to happen for the first sentence to be false, one realizes that all that is needed is that at least one number in \( A \) should be even. So in fact the negation of the sentence is, “There exists a number in \( A \) that is even.”

What explains the temptation to give the first, incorrect answer? One possibility emerges when one writes the sentence more formally, thus:

\[
(11) \quad \forall n \in A \quad n \text{ is odd.}
\]

The first answer is obtained if one negates just the last part of this sentence, “\( n \) is odd”; but what is asked for is the negation of the whole sentence. That is, what is wanted is not

\[
(12) \quad \forall n \in A \quad \neg(n \text{ is odd}),
\]

but rather

\[
(13) \quad \neg(\forall n \in A \quad n \text{ is odd}),
\]

which is equivalent to

\[
(14) \quad \exists n \in A \quad n \text{ is even.}
\]

A second possible explanation is that one is inclined (for psycholinguistic reasons) to think of the phrase “every element of \( A \)” as denoting something like a single, typical element of \( A \). If that comes to have the feel of a particular number \( n \), then we may feel that the negation of “\( n \) is odd” is “\( n \) is even.” The remedy is not to think of the phrase “every element of \( A \)” on its own: it should always be part of the longer phrase, “for every element of \( A \).”

3.4 Free and Bound Variables

Suppose we say something like, “At time \( t \) the speed of the projectile is \( v \).” The letters \( t \) and \( v \) stand for real numbers, and they are called variables, because in the back of our mind is the idea that they are changing. More generally, a variable is any letter used to stand for a mathematical object, whether or not one thinks of that object as changing through time. Let us look once again at the formal sentence that said that a positive integer \( m \) is prime:

\[
(10) \quad \forall a, b \quad ab = m \Rightarrow ((a = 1) \lor (b = 1)).
\]

In this sentence, there are three variables, \( a, b, \) and \( m \), but there is a very important grammatical and semantic difference between the first two and the third. Here are two results of that difference. First, the sentence does not really make sense unless we already know what \( m \) is from the context, whereas it is important that \( a \) and \( b \) do not have any prior meaning. Second, while it makes perfect sense to ask, “For which values of \( m \) is sentence (10) true?” it makes no sense at all to ask, “For which values of \( a \) is sentence (10) true?” The letter \( m \) in sentence (10) stands for a fixed number, not specified in this sentence, while the letters \( a \) and \( b \), because of the initial \( \forall a, b \), do not stand for numbers—rather, in some way they search through all pairs of positive integers, trying to find a pair that multiply together to give \( m \). Another sign of the difference is that you can ask, “What number is \( m \)” but not, “What number is \( a \)” A fourth sign is that the meaning of sentence (10) is completely unaffected if one uses different letters for \( a \) and \( b \), as in the reformulation

\[
(10') \quad \forall c, d \quad cd = m \Rightarrow ((c = 1) \lor (d = 1)).
\]

One cannot, however, change \( m \) to \( n \) without establishing first that \( n \) denotes the same integer as \( m \). A variable such as \( m \), which denotes a specific object, is called a free variable. It sort of hovers there, free to take any value. A variable like \( a \) and \( b \), of the kind that does not denote a specific object, is called a bound variable, or sometimes a dummy variable. (The word “bound” is used mainly when the variable appears just after a quantifier, as in sentence (10).)

Yet another indication that a variable is a dummy variable is when the sentence in which it occurs can be rewritten without it. For instance, the expression \( \sum_{n=1}^{100} f(n) \) is shorthand for \( f(1) + f(2) + \cdots + f(100) \), and the second way of writing it does not involve the letter \( n \), so \( n \) was not really standing for anything in
I. Introduction

the first way. Sometimes, actual elimination is not possible, but one feels it could be done in principle. For instance, the sentence “For every real number \( x \), \( x \) is either positive, negative, or zero” is a bit like putting together infinitely many sentences such as “\( t \) is either positive, negative, or zero,” one for each real number \( t \), none of which involves a variable.

4 Levels of Formality

It is a surprising fact that a small number of set-theoretic concepts and logical terms can be used to provide a precise language that is versatile enough to express all the statements of ordinary mathematics. There are some technicalities to sort out, but even these can often be avoided if one allows not just sets but also numbers as basic objects. However, if you look at a well-written mathematics paper, then much of it will be written not in symbolic language peppered with symbols such as \( \forall \) and \( \exists \), but in what appears to be ordinary English. (Some papers are written in other languages, particularly French, but English has established itself as the international language of mathematics.)

How can mathematicians be confident that this ordinary English does not lead to confusion, ambiguity, and even incorrectness?

The answer is that the language typically used is a careful compromise between fully colloquial English, which would indeed run the risk of being unacceptably imprecise, and fully formal symbolism, which would be a nightmare to read. The ideal is to write in as friendly and approachable a way as possible, while making sure that the reader (who, one assumes, has plenty of experience and training in how to read mathematics) can see easily how what one writes could be made more formal if it became important to do so. And sometimes it does become important: when an argument is difficult to grasp it may be that the only way to convince oneself that it is correct is to rewrite it more formally.

Consider, for example, the following reformulation of the principle of mathematical induction, which underlies many proofs:

\[
(15) \text{Every nonempty set of positive integers has a least element.}
\]

If we wish to translate this into a more formal language we need to strip it of words and phrases such as “nonempty” and “has.” But this is easily done. To say that a set \( A \) of positive integers is nonempty is simply to say that there is a positive integer that belongs to \( A \). This can be stated symbolically:

\[
(16) \exists n \in \mathbb{N} \quad n \in A.
\]

What does it mean to say that \( A \) has a least element? It means that there exists an element \( x \) of \( A \) such that every element \( y \) of \( A \) is either greater than \( x \) or equal to \( x \) itself. This formulation is again ready to be translated into symbols:

\[
(17) \exists x \in A \quad \forall y \in A \quad (y > x) \lor (y = x).
\]

Statement (15) says that (16) implies (17) for every set \( A \) of positive integers. Thus, it can be written symbolically as follows:

\[
(18) \forall A \subset \mathbb{N} \\
[ (\exists n \in \mathbb{N} \quad n \in A) \Rightarrow (\exists x \in A \quad \forall y \in A \quad (y > x) \lor (y = x)) ].
\]

Here we have two very different modes of presentation of the same mathematical fact. Obviously (15) is much easier to understand than (18). But if, for example, one is concerned with the foundations of mathematics, or wishes to write a computer program that checks the correctness of proofs, then it is better to work with a greatly pared-down grammar and vocabulary, and then (18) has the advantage. In practice, there are many different levels of formality, and mathematicians are adept at switching between them. It is this that makes it possible to feel completely confident in the correctness of a mathematical argument even when it is not presented in the manner of (18)—though it is also this that allows mistakes to slip through the net from time to time.

I.3 Some Fundamental Mathematical Definitions

The concepts discussed in this article occur throughout so much of modern mathematics that it would be inappropriate to discuss them in part III— they are too basic. Many later articles will assume at least some acquaintance with these concepts, so if you have not met them, then reading this article will help you to understand significantly more of the book.

1 The Main Number Systems

Almost always, the first mathematical concept that a child is exposed to is the idea of numbers, and numbers retain a central place in mathematics at all levels.
Learning the Language of Mathematics

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Just as everybody must strive to learn language and writing before he can use them freely for expression of his thoughts, here too there is only one way to escape the weight of formulas. It is to acquire such power over the tool that, unhampered by formal technique, one can turn to the true problems.

— Hermann Weyl [4]

This paper is about the use of language as a tool for teaching mathematical concepts. In it, I want to show how making the syntactical and rhetorical structure of mathematical language clear and explicit to students can increase their understanding of fundamental mathematical concepts. I confess that my original motivation was partly self-defense: I wanted to reduce the number of vague, indefinite explanations on homework and tests, thereby making them easier to grade. But I have since found that language can be a major pedagogical tool. Once students understand HOW things are said, they can better understand WHAT is being said, and only then do they have a chance to know WHY it is said. Regrettably, many people see mathematics only as a collection of arcane rules for manipulating bizarre symbols — something far removed from speech and writing. Probably this results from the fact that most elementary mathematics courses — arithmetic in elementary school, algebra and trigonometry in high school, and calculus in college — are procedural courses focusing on techniques for working with numbers, symbols, and equations. Although this formal technique is important, formulae are not ends in themselves but derive their real importance only as vehicles for expression of deeper mathematical thoughts. More advanced courses — such as geometry, discrete mathematics, and abstract algebra — are concerned not just with manipulating symbols and solving equations but with understanding the interrelationships among a whole host of sophisticated concepts. The patterns and relationships among these concepts

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constitute the “true problems” of mathematics. Just as procedural mathematics courses tend to focus on “plug and chug” with an emphasis on symbolic manipulation, so conceptual mathematics courses focus on proof and argument with an emphasis on correct, clear, and concise expression of ideas. This is a difficult but crucial leap for students to make in transitioning from rudimentary to advanced mathematical thinking. At this stage, the classical trivium of grammar, logic, and rhetoric becomes an essential ally.

There is, in fact, a nearly universally accepted logical and rhetorical structure to mathematical exposition. For over two millennia serious mathematics has been presented following a format of definition-theorem-proof. Euclid’s *Elements* from circa 300 BC codified this mode of presentation which, with minor variations in style, is still used today in journal articles and advanced texts. There is a definite rhetorical structure to each of these three main elements: definitions, theorems, and proofs. For the most part, this structure can be traced back to the Greeks, who in their writing explicitly described these structures. Unfortunately, this structure is often taught today by a kind of osmosis. Fragmented examples are presented in lectures and elementary texts. Over a number of years, talented students may finally unconsciously piece it all together and go on to graduate school. But the majority of students give up in despair and conclude that mathematics is just mystical gibberish.

With the initial support of a grant from Clemson’s Pearce Center for Technical Communication and the long-term moral support of the Communication Across the Curriculum program, I have been working for several years now on developing teaching strategies and developing teaching materials for making the syntactical and logical structure of mathematical writing clear and explicit to students new to advanced mathematics. The results have been gratifying: if the rules of the game are made explicit, students can and will learn them and use them as tools to understand abstract mathematical concepts. Several years ago, I had the opportunity of sharing these ideas with the Occasional Seminar on Mathematics Education at Cornell, and now through this paper, I hope to share them with a wider audience.

*One should NOT aim at being possible to understand, but at being IMPOSSIBLE to misunderstand.*

— Quintilian, circa 100 AD

The use of language in mathematics differs from the language of ordinary speech in three important ways. First it is nontemporal — there is no past, present, or future in mathematics. Everything just “is”. This presents difficulties in forming convincing examples of, say, logical prin-
Learning the Language of Mathematics

ciples using ordinary subjects, but it is not a major difficulty for the student. Also, mathematical language is devoid of emotional content, although informally mathematicians tend to enliven their speech with phrases like “Look at the subspace killed by this operator” or “We want to increase the number of good edges in the coloring.” Again, the absence of emotion from formal mathematical discourse or its introduction in informal discourse presents no difficulty for students.

The third feature that distinguishes mathematical from ordinary language, one which causes enormous difficulties for students, is its precision. Ordinary speech is full of ambiguities, innuendoes, hidden agendas, and unspoken cultural assumptions. Paradoxically, the very clarity and lack of ambiguity in mathematics is actually a stumbling block for the neophyte. Being conditioned to resolving ambiguities in ordinary speech, many students are constantly searching for the hidden assumptions in mathematical assertions. But there are none, so inevitably they end up changing the stated meaning — and creating a misunderstanding. Conversely, since ordinary speech tolerates so much ambiguity, most students have little practice in forming clear, precise sentences and often lack the patience to do so. Like Benjamin Franklin they seem to feel that mathematicians spend too much time “distinguishing upon trifles to the disruption of all true conversation.”

But this is the price that must be paid to enter a new discourse community. Ambiguities can be tolerated only when there is a shared base of experiences and assumptions. There are two options: to leave the students in the dark, or to tell them the rules of the game. The latter involves providing the experiences and explaining the assumptions upon which the mathematical community bases its discourse. It requires painstaking study of details that, once grasped, pass naturally into the routine, just as a foreign language student must give meticulous attention to declensions and conjugations so that he can use them later without consciously thinking of them. The learning tools are the same as those in a language class: writing, speaking, listening, memorizing models, and learning the history and culture. Just as one cannot read literature without understanding the language, similarly in mathematics (where “translation” is not possible) this exacting preparation is needed before one can turn to the true problems. Thus it has become an important part of all my introductory courses, both at the undergraduate and graduate level.

This paper is a report on my efforts to make the rhetorical and syntactical structure of mathematical discourse explicit and apparent to the ordinary student. For concreteness sake, it is based on examples from a College Geometry course for juniors majoring in Secondary Mathematics Education. The same principles and goals apply, however, from freshman discrete mathematics for computer science majors to the linear algebra
course for beginning math graduate students. As such it is about teaching and learning the tool of language in mathematics and not about grappling with the deeper problems such as the discovery of new mathematics or the heuristic exposition of complex mathematical ideas or the emotional experience of doing mathematics. As important as these deeper problems are, they cannot be approached without first having power over the tool of language. Mastering the trivium is necessary before the quadrivium can be approached.

*Mathematics cannot be learned without being understood*  
— *it is not a matter of formulae being committed to memory*  
*but of acquiring a capacity for systematic thought.*  
— Peter Hilton [3]

Systematic thought does not mean reducing everything to symbols and equations — even when that is possible. Systematic thought also requires precise verbal expression. Since serious mathematics is usually communicated in the definition-theorem-proof format, the first step in learning the formal communication of mathematics is in learning definitions. For this reason, and because it requires the least technical sophistication, I will illustrate my general methodology with definitions. Although the examples below are kept elementary for the sake of the general reader, the principles they illustrate become even more critical the more advanced the material. This is sometimes a difficult point for students, who may not understand the need for meticulous precision with elementary concepts. But to have the technique needed to deal with complicated definitions, say the definitions of equivalence relations or of continuity, it is necessary to first practice with simple examples like the definition of a square.

Let us begin with a definition of definitions and some examples of good and bad definitions. A definition is a *concise* statement of the *basic* properties of an object or concept which *unambiguously identify* that object or concept. The italicized words give the essential characteristics of a good definition. It should be concise and not ramble on with extraneous or unnecessary information. It should involve basic properties, ideally those that are simply stated and have immediate intuitive appeal. It should not involve properties that require extensive derivation or are hard to work with. In order to be complete, a definition must describe exactly the thing being defined — nothing more, and nothing less.

**GOOD DEFINITION:** A rectangle is a *quadrilateral* all four of whose angles are right angles.
POOR DEFINITION: A rectangle is a parallelogram in which the diagonals have the same length and all the angles are right angles. It can be inscribed in a circle and its area is given by the product of two adjacent sides.

This is not CONCISE. It contains too much information, all of which is correct but most of which is unnecessary.

POOR DEFINITION: A rectangle is a parallelogram whose diagonals have equal lengths.

This statement is true and concise, but the defining property is not BASIC. This would work better as a theorem to be proved than as a definition. In mathematics, assertions of this kind are regarded as characterizations rather than as definitions.

BAD DEFINITION: A rectangle is a quadrilateral with right angles.

This is AMBIGUOUS. With some right angles? With all right angles? There are lots of quadrilaterals that have some right angles but are not rectangles.

UNACCEPTABLE DEFINITION: rectangle: has right angles

This is unacceptable because mathematics is written as English is written — in complete, grammatical sentences. Such abbreviations frequently hide major misunderstandings as will be pointed out below.

In Aristotle’s theory of definition, every “concept is defined as a subclass of a more general concept. This general concept is called the genus proximum. Each special subclass of the genus proximum is characterized by special features called the differentiae specificae.” [1, p. 135] We will refer to these simply as the genus and species. In each example above, the italicized word is the genus. In the case of rectangle, the genus is the class of quadrilaterals and the species is the requirement that all angles be right angles. One of the greatest difficulties students experience with new concepts is that they fail to understand exactly what the genus is to which the concept applies. The unacceptable definition above skirts this issue by avoiding the genus altogether. To illustrate the importance of genus, note that we cannot say:
These two points are parallel.
This triangle is parallel.
The function \( f(x) = 3x + 1 \) is parallel.
35 is a parallel number.

The term “parallel” has as its genus the class of pairs of lines (or more generally, pairs of curves). Any attempt to apply the word “parallel” to other kinds of objects, like pairs of points, triangles, functions, or numbers, results not in a “wrong” statement but in nonsense. Note that the nonsense is not grammatical, but rhetorical. The four statements above are all perfectly grammatical English sentences, but none of them makes sense because of the inappropriate genus. Students only rarely make nonsensical statements like the four above because the genus is on a sufficiently concrete level that confusion is unlikely. However, when several layers of abstraction are superimposed, as is common in modern mathematics, nonsense statements become more common. Let us look at a specific abstract example.

In geometry parallelism, congruence, and similarity are all examples of the general notion of an equivalence relation. Equivalence relations abstract the basic properties of “sameness” or equality — for example, similar triangles have the same shape and parallel lines have equal slopes. Euclid includes one such property of equivalence relations as the first of his common notions: “Things which are equal to the same thing are also equal to one other.” [3] In modern terms, this property is called “transitivity” and is enunciated formally as follows:

A relation \( R \) on a set \( X \) is transitive if and only if for all choices of three elements \( a, b, \) and \( c \) from \( X \), if \( a \) is related to \( b \) and \( b \) is related to \( c \), then \( a \) must also be related to \( c \).

Let us look at this definition from the standpoints of rhetoric, grammar, and logic. Rhetorically, there are three layers of abstraction in this definition: first, the objects or elements (which are abstract rather than definite), then the set \( X \) of such objects, and finally the relation \( R \) on this set. Students struggling with these layers of abstraction tend to get them confused and may say:

“\( a, b, \) and \( c \) are not transitive but \( e, f, \) and \( g \) are.”
“\( \text{The set } X \text{ is transitive.} \)”

Such statements do not make sense because they attempt to apply the term “transitive” at a lower layer of abstraction than its genus requires. Although it may be possible to guess what the student has in mind, it is
important to stress that this is not enough, as the Quintilian quote empha-
sizes.

The definition of transitivity also illustrates the absence of ambigu-
ity. There is no hidden assumption that \(a\) is related to \(b\). There is no
hidden assumption that \(a\) and \(c\) must be different. These assumptions are
not left up to the discretion of the student or the whim of the professor.
They are simply not there. Yet these assumptions are often tacitly made
by students trying to understand transitivity.

Grammatically, students have a tendency to use the active voice “a
relates to \(b\)” rather than the passive “\(a\) is related to \(b\)”, which is standard
mathematical usage. Attention to this single, simple linguistic detail seems
to heighten the focus on listening for proper usage and as a consequence
proper understanding. Students who are attentive and disciplined enough
to pick up this minor detail, which incidentally I repeatedly stress, gener-
ally are more secure with the concepts and more likely to apply them
correctly. Shallow listening leads to shallow understanding. Here the
difference is not a significant one conceptually, but it is a difference which
is universal in the culture of mathematical discourse and thus is a shibbo-
leth for distinguishing a “native speaker” from an outsider.

Of course, understanding the definition of transitivity also requires
understanding the logical structure of the species. In this case, the spe-
cies involves two logical connectives: AND (logical conjunction) and IF
... THEN (implication) preceded by a universal quantifier FOR ALL. All of
these present major difficulties for many students due to the comparative
sloppiness of ordinary speech. For example, “any” is an ambiguous word
since it can be used in both the universal and existential senses:

\[
\begin{align*}
\text{Can anyone work this problem?} & \quad \text{(existential quantifier)} \\
\text{Anyone can do it!} & \quad \text{(universal quantifier)}
\end{align*}
\]

For this reason I urge students to avoid the use of “any” when
trying to learn the use of quantifiers. Although much more could be said
on these issues, for brevity let me turn immediately to the one which is by
far most important and most difficult: implication.

Implications are the backbone of mathematical structure. Many
definitions (like transitivity) involve implications and almost all theorems
are implications with a hypothesis and a conclusion. Like the Eskimo
“snow,” the phenomenon is so pervasive in mathematical culture that we
have evolved many different ways of expressing it. Here are eight differ-
ent but equivalent ways of stating that squares are rectangles, with names
for some of the variations given on the side:
1) If a figure is a square, then it is a rectangle.  Hypothetical
2) A figure is a square only if it is a rectangle.
3) A figure is a rectangle whenever it is a square.
4) All squares are rectangles.  Categorical
5) For a figure to be a square, it must necessarily be a rectangle.  Necessity
6) A sufficient condition for a figure to be a rectangle is that it be a square.  Sufficiency
7) A figure cannot be a square and fail to be a rectangle.  Conjunctive
8) A figure is either a rectangle or it is not a square.  Disjunctive

There are three major issues involved in understanding implications. Two of these are purely logical:

1) realizing that an implication is not the same as a conjunction:

“If quadrilateral ABCD is a square, then it is a rectangle.”

    is not the same as

“Quadrilateral ABCD is a square and a rectangle.”

2) realizing that an implication is not the same as its converse:

“If quadrilateral ABCD is a square, then it is a rectangle.”

    is not the same as

“If quadrilateral ABCD is a rectangle, then it is a square.”

The third issue is a more subtle rhetorical issue involving a grasp of the relationship between premise and conclusion. The relationship is not one of causality, and the premise and conclusion can be implicit in a turn of phrase that is not an explicit if-then statement. An excellent exercise is to give students a dozen or so implications, expressed in different ways, and ask them to find the premise and conclusion in each. Then ask them to reformulate each implication in several different ways, just as I did above for “Squares are rectangles.” It is not necessary, and in fact in some ways undesirable, for the students to understand the meaning of the statements. The point here is that these are syntactical exercises, and it is enough to have a feel for the language and an understanding of syntax to be successful. It does not depend on the actual meaning. At this point as
in the learning of definitions, I stress that the results must read and sound like good English sentences.

How is all of this implemented in the classroom? As I said above, I proceed similarly to teaching a foreign language. Early in the semester, I present the students with a list of roughly twenty common geometrical terms, such as, circle, square, trapezoid and midpoint, and for homework ask them to write out definitions. I provide them with the following “Guidelines for Definitions in Good Form”:

1. A definition MUST be written as a complete, grammatically correct English sentence.
2. A definition MUST be an “if and only if” statement.
3. A definition MUST have a clearly stated genus and a clearly stated species.
4. The quantifiers in a good definition MUST be explicitly and clearly stated.
5. The term being defined MUST be underlined.

The next few class periods are spent with students putting their definitions on the board. The class and I critique them according to the principles outlined above. This invariably brings to the fore many issues, ranging from a reluctance to write in complete sentences and a decided preference for symbols over words to the syntactical issues described above. Many misconceptions can be brought to light and usually corrected. I also call on students to state definitions verbally. By engaging both speaking and writing, I hope to more deeply and actively penetrate the students’ thinking.

We also explore the meaning of the definitions, the range of choices available, and some of the history involved. For example, Aristotle (384 - 322 BC) insisted that the subclasses (species) of each genus be disjoint: they could not overlap and one subclass could not include another. Thus for Aristotle, a square was NOT a rectangle. [1, p. 136] From the modern point of view this is inconvenient. Virtually everything one wants to prove about non-square rectangles also holds for squares, so it is a nuisance to have to state and prove two separate theorems. The modern standard is that squares are special cases of rectangles, so theorems about rectangles also apply to squares.

Finally, students are assigned to groups, first to provide feedback on the members’ definitions and later to compile as a group a list of “standard” definitions in good form for all the given terms.

I do not require students to memorize common geometric definitions, but when we reach the abstraction of transitivity and equivalence relations, I provide models which must be memorized. There are two main
reasons for this. First, it is not possible to have a good class discussion involving these concepts if students must constantly flip through their notes to look up the definitions. Second, the definitions I provide are models of good mathematical expression, something which is often lacking in elementary texts. Students can use these models to help build their own definitions (and later, theorems and proofs), but most importantly, repeating them out loud and memorizing them helps develop an ear for how correct mathematical discourse should sound.

ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗ ΕΙΣΙΤΩ

“Let no one ignorant of geometry enter here”
— Plato, now the Motto of the American Mathematical Society

In conclusion, I want to confess what my real goals are in teaching this material. In a society in which information is passed in 60 second sound bites and reasoning limited to monosyllabic simple sentences, careful, analytic thinking is in danger of extinction. And this is a grave danger in a democratic society beset by a host of very complex moral and social problems. When geometry passed from the pragmatic, monarchical Egyptian surveyors to the democratic Greek philosophers nearly three millennia ago, its purpose changed. True, geometry (and more generally mathematics) has been many practical applications. But that is not why geometry has retained a universal place in the curriculum. It has been taught to teach reasoning and intellectual discipline. This why Plato placed his famous motto over the academy door. That is why Abraham Lincoln studied Euclid. And that remains my main goal in teaching.

Notes

Building and Modeling How to Build a Community of Learners

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Abstract

This article describes how to build a classroom community of learners, pre-service teachers, during the first days of the semester. This process not only creates a supportive learning environment, but it provides pre-service teachers with a model of how they can build a community of learners in their own classrooms.

Introduction

In the past, when I taught mathematics courses for preservice teachers, I struggled to create a classroom environment where even the shyest students shared and justified their ideas, took risks, and challenged each other’s thoughts and work. To realize the full potential of the mathematics learning embedded in tasks, I knew a classroom atmosphere that supported mathematical inquiry, risk taking, and higher order mathematical discourse needed to be created (Erickson 1999; Marcus & Fey 2003; Rasmussen, Yackel and King 2003). Having students work in groups on worthwhile tasks, modeling good questioning, and providing support as a facilitator sometimes was enough to create this environment. However, the semester was often well underway before the class resembled a learning community, and I could step back from my role as cheerleader for the environment. I wanted to create a collaborative atmosphere from the first day of my courses, but thought it should happen organically. After some reflection and insight, I learned a valuable lesson. If I wanted to create a community of learners at the onset of a course, I needed to dedicate time at the beginning to build the foundation for this environment; classroom learning communities don’t often cultivate quickly organically. The article that follows describes how I intentionally build a supportive classroom community of learners during the initial days of the semester. This process not only helps to create a supportive learning environment, but it provides the preservice teachers with a model of how they can build a community of learners in their own classrooms.

Creating Course Norms

Each semester, I begin with an activity developed to create course norms. Because they are the class’s norms and not my norms as a facilitator, I believe the preservice teachers should participate in the process of creating the norms and have ownership of them. To help the students consider what they need from each other in order to feel supported and comfortable to take risks and share their ideas, I ask them the following questions: “What do you want others to know about you as a learner of mathematics? What supports you when you are doing mathematics in a group? What do you need from others in order to share your ideas? What do you not want others to say or do?” I provide examples of what I consider essential norms, like maintaining a positive attitude,
allowing everyone to share their ideas, and respecting others’ thoughts and ideas. Each student is then given small sheets of stick-on notes to record their responses to these questions, writing one response per sheet. After having time to reflect, the students share their responses in small groups of three to five, which provides them a safe platform to discuss their fears, needs as learners, and beliefs about mathematics and learning mathematics. The stick-on notes allow them to join those similar or shared responses. The groups then report out, and as a class, we use the responses to discuss and create our list of classroom norms. The shared fear of mathematics and of being wrong always arises as does the need to give others sufficient time to think about a problem or question before sharing their ideas. At the following class, I give the students a sheet that contains our norms, and we begin this class by having each person (students and instructor) read the norms out loud. I then discuss the importance of maintaining the norms and allowing everyone to feel supported and safe, so we can do our best work as learners of mathematics. I also note that this list shouldn’t be put in their folders and forgotten. To be true to it and ourselves as learners, we have to reflect on it often and continue to evaluate how well we are maintaining our norms.

Community-Building Mathematical Tasks

After creating our norms, we are ready to jump in and do some math! My goal for the first few tasks I present to the students each semester goes beyond having the students doing mathematics (Stein & Smith 1998) and developing a deeper understanding of the mathematics and focuses on communicating about mathematics with each other. An ideal task promotes both mathematical thinking and communication among all of the students. For my number systems course, I use an activity, adapted from Bresser and Holtzman’s (2006) “Guess My Number” in Minilessons for Math Practice, that I call “Guess Your Number?” I stick one number on each student’s back (e.g. \(\frac{3}{4}\), \(\pi\), 0, -5, 2.1). They are then given the directions to ask everyone in the class one yes-or-no question to gather information about the number with the end goal of deducing the number on their back. Because the course focuses on the real number system and its subsystems, this activity provides a great gateway into the mathematics of the course. Halfway through this activity, I stop the students, and we discuss their questions and what makes a question a good one for this activity. Usually someone will introduce the idea that the questions that eliminate large groups or sets of numbers are the most helpful and then give an example, such as asking if it is an odd number, an even number, positive, or negative. Most of the students follow the discussion with more thoughtful and intentional questions. Beyond thinking about the mathematics, each student in the course discusses mathematics with every other student in the course and becomes comfortable communicating about mathematics with the others. Even though the task is simple, it is quite beneficial in terms of building a community of mathematics learners.

Another task, Math Meetings (Driscoll 2001), that I often use in my algebra course is adapted from Driscoll’s professional development materials on fostering algebraic thinking. It is a variation of the common Handshake Problem. To introduce this task, I give the students the following directions:

Each person in the room is going to talk with everyone else one person at a time. You are to find and record one new shared math fact that is true for both people. Shared math facts encompass a lot. For example, one shared fact could be that you both
really like geometry or that you both are observing in a seventh-grade math classroom. After you have recorded your math fact, find a new person and discuss your shared math fact. You are not to reuse facts. For example, Maddy can’t use the fact that she enjoys math with two different people.

After everyone has discussed their math facts with one another, I then pose the following extension:

Each person in the room should have talked with everyone else and discussed and recorded their shared math facts. How many distinct facts have been recorded all together as a whole class? For example, if Riley and Carter recorded that they both like algebra, and Eric and Cole recorded the same fact about themselves, consider these facts to be distinct because they are about different people.

The students then work in groups of two to three to come up with a solution to this problem. The discussion that follows focuses on the numerous ways to solve it and on understanding and making connections to each other’s methods and ways of thinking. This task allows the students to experience algebraic thinking, which is a great introduction to the course. Moreover, during this task, the class begins to build a sense of trust and community, and students are given another chance to discuss their beliefs and experiences about mathematics teaching and learning with their peers.

The third community-building task, Getting to Know Colleagues: Uniquely Us, is taken from Rubenstein, Beckman, & Thompson’s (2004) *Teaching and Learning Middle Grades Mathematics*. This task is perfect for a preservice teacher course focusing on data analysis. In groups of three or four, students collect information about each other. The following lists some of the suggested information: mathematics classes taken in high school or college, movies watched about mathematics, experiences working with middle grades students, other teaching or field experiences, experiences tutoring, shared beliefs about mathematics, etc. I recommend that the students focus on topics related to mathematics and teaching and learning mathematics. They are asked to find and record unique and interesting ways that the group members are alike and different and then to use this information to create a Venn diagram with intersecting circles, one for each member of the group, to present to the rest of the class. A brief review of Venn diagrams is often needed. When each group presents their diagram to the entire class, each person discusses how they were different from the other members of their group, so that everyone in the class is required to share their uniqueness. Again, the beauty of this task is that it initiates a sense of sharing and respect for everyone’s ideas while still maintaining a high level of mathematics. Students are not only doing mathematics (many often come to the realization that a Venn diagram is a meaningful way to display data for the first time ever); all students share and discuss their ideas from the start.

**Concluding Thoughts**

Creating classroom norms together, discussing students’ fears of mathematics, and intentionally choosing inclusive tasks that allow all students to communicate about mathematics are effective ways to build a supportive, powerful learning environment that encourages students to take risks and share their ideas, conjectures, and insights without fear of ridicule or embarrassment. Moreover, continuing to model effective practices is essential to maintaining this safe environment. Having students explain their thinking, asking a student to paraphrase another’s insights or build on their ideas, making sure
everyone is justifying ideas with mathematical reasoning, valuing mistakes and misconceptions as important learning opportunities, and reminding students of the course norms are all ways to maintain this mathematical learning community. Modeling, reflecting on, and discussing ways to create this supportive learning environment are also essential aspects of a mathematics course for preservice teachers who one day are going to cultivate such an atmosphere of mutual respect in their own classrooms.

References