Solutions for common final exam questions

Math 117 – Fall 2021

1.2 Rate of change

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(2)-f(1)}{2-1} = \frac{13-4}{1} = 9$

   (b) $\frac{\Delta f}{\Delta x} = \frac{n-k}{m-j}$

   (c) $\frac{\Delta f}{\Delta x} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$

2. The correct answer is (d) All of the above.

3. The correct choice is (b).

4. $\frac{\Delta f}{\Delta x} = \frac{f(90)-f(25)}{90-25} \approx \frac{6.5-5}{90-25} = \frac{1.5}{65} \approx 0.02$

1.3 Linear functions

1. (a) Yes, this could be a linear function with rate of change $\frac{\Delta f}{\Delta x} = 2$.

   (b) This could not be a linear function. It does not have a constant rate of change.

2. The intercept is 54.25 and the slope is $-\frac{2}{7}$. The town had a population of 54,250 in 1970 and it decreased at a rate of $\frac{2}{7}$ thousand people, or about 285 people, per year.

3. The intercept is 17.75 and the slope is $\frac{1}{250}$. The stalactite measured 17.75 inches when first measured and has grown at a constant rate of $\frac{1}{250} = 0.004$ inch per year since then.

4. (a) $y = 300 + \frac{1}{250}x$.

   (b) If $x = 25,000$ then $y = 400$ and if $x = 50,000$ then $y = 600$.

   (c) Solve $700 = 200 + \frac{1}{250}x$ to get $x = 100,000$ dollars.

   (d) The slope is $\frac{20}{5000} = \frac{1}{250}$ units per dollar. Each dollar spend on advertising increases sales by $\frac{1}{250}$ unit.

5. $g(x) = x + 1$ and $h(x) = 4 - x$, so $f(x) = (x+1) - (4-x) = 2x - 3$. The y intercept of f is $f(0) = -3$ and the x intercept is the solution to $f(x) = 0$, namely $x + \frac{3}{2}$.

1.4 Formulas for linear functions

1. (a) $y = -4x + 7$

   (b) $y = -2x + 3$

   (c) $y = \frac{2}{3}x + \frac{11}{x}$.

   (d) $f(x) = -2.4x + 8$

   (e) $y = x + 6$

2. The parallel line is $y = -4x + 9$ and the perpendicular line is $y = \frac{1}{4}x + \frac{19}{4}$.

3. (a) $C(175) = 11375$
   (b) $C(175) - C(150) = 11375 - 11250 = 125$
   (c) $\dfrac{C(175) - C(150)}{175 - 150} = \frac{125}{25} = 5$
   (d) Using the slope we've computed $C(0) = C(100) - 5 \times 100 = 11000 - 500 = 10500$. There is a fixed cost of \$10,500 before any goods are produced.
   (e) $C(x) = 10\,500 + 5n$.

4. (a) The slope is $frac\Delta q \Delta p = \frac{65-45}{3.10-3.50} = \frac{20}{-0.4} = -50$, and we can solve for the intercept to find $q = -50p + 220$.
   (b) The demand for gasoline falls at a rate of 50 gallons per dollar of price increase.
   (c) The q intercept is 220, meaning that the maximum demand for gasoline would be 220 gallons if it were free.
   (d) The p intercept is the value of p when $q = 0$. That is $p = 4.4$, meaning that demand will fall to zero when the price is \$4.40 per gallon.

5. The slope is $\frac{\Delta f}{\Delta x} = \frac{-18-17}{4-(-3)} = \frac{-35}{7} = -5$ and we can solve for the intercept to get $f(x) = -5x + 2$. The completed table is

x	-3	0	$\frac{1}{5}$	4	7	$\frac{32}{5}$
$f(x)$	17	2	1	-18	-33	-30

6. Using the function values in the table we can determine the following slopes:

$$\frac{\Delta r}{\Delta x} = 2 \quad \frac{\Delta s}{\Delta x} = -2 \quad \frac{\Delta t}{\Delta x} = -\frac{1}{2} \quad \frac{\Delta u}{\Delta x} = 2.$$

   This means that r and u are parallel and s is perpendicular to both of these.

7. We want $-\frac{2}{a} = -\frac{1}{3}$, so $a = 6$.

8. The correct choice is (d).

9. $b = 2$ and $a = 1$.

1.5 Modeling with linear functions

1. Matches for graphs are shown in the grid below. There are no graphs matching equations (c) or (g), and one graph matching no equation.

   (a) We can write linear functions for each. The value of the Frigbox is $F(t) = 950 - 50t$ and the value of the Arctic Air is $A(t) = 1200 - 100t$ after t years. They are equal when $950 - 50t = 1200 - 100t$, or after $t = 5$ years.
   (b) $F(20) = -50$ and $A(20) = -800$. The most reasonable interpretation is that by this time both refrigerators' value has depreciated to zero.

2. l_1 has slope $-\frac{2}{3}$, and so l_2 is given by $y = \frac{3}{2}x$.

3. P has coordinates $(1, 0)$.

2.1 Input and output

1. (a) $s(2) = 146$. The car is at mile 146.
   (b) $v(t) = 65$.
   (c) Solve $v(t) = 67$ to get $t = 3$. At this time the car's position is $s(3) = 202$ miles.

2. (a) $(P0) = -500$. If the theater sells no tickets, they incur a \$500 loss.
   (b) Profit will equal zero then $P(n) = 0$, or when the theater sells $n = 25$ tickets.
   (c) $P(100)$ is the theater's profit when they sell 100 tickets. It's units are dollars.

3. (a) $s(2) - s(1) < 0$
   (b) $s(3) - s1) = 0$
   (c) $s(4) - s(3) > 0$
   (d) $s(1) - S(4) < 0$

4. (a) B
   (b) A
   (c) D
   (d) E
   (e) C

5. (a) $g(-25x) = 625x - 25x$
   (b) $g(25 - x) = (25 - x)^2 + (x - 25) = 600 - 49x + x^2$
   (c) $g(x + \pi) = (x + \pi)^2 + (x + \pi) = x^2 + 2\pi x + \pi^2 + x + \pi$
   (d) $g(\sqrt{x}) = x + \sqrt{x}$
   (e) $g\left(\frac{9}{x+1}\right) = \left(\frac{9}{x+1}\right)^2 + \frac{9}{x+1} = \frac{81}{(x+1)^2} + \frac{9}{x+1} = \frac{90+9x}{(x+1)^2}$
   (f) $g(x^2) = x^4 + x^2$

6. (a) $k(-2) = 8 - (-2)^2 = 12$, so $(-2, 12)$ is on the graph of k.
   (b) Solve $k(x) = -24$ to get $x = \pm\sqrt{32}$ or $x = \pm 2\sqrt{2}$. The two points on the graph of k are $(2\sqrt{2}, -24)$ and $(-2\sqrt{2}, -24)$.

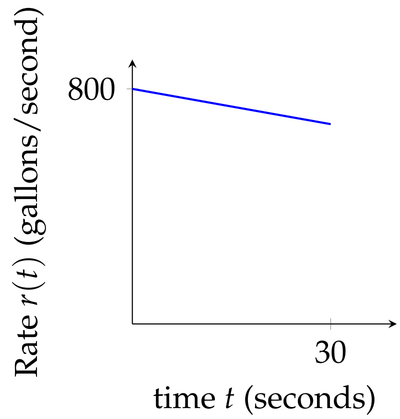
2.2 Domain and range

1. The function graphed on the left has domain $0 \leq x \leq 4$ and range $0 \leq y \leq 2$. The one graphed on the right has domain $1 \leq x \leq 5$ and range $1 \leq y \leq 6$

2. (a) The domain is all real numbers $t < -2$ or $t > 2$.
   (b) The domain is all real numbers $x \geq -9$.
   (c) The domain is all real numbers $x \leq -6$ or $x \geq 6$.

3. (a) $r(0) = 800$, $r(15) = 740$, and $r(25) = 700$. This means that at time 0, water enters the reservoir at a rate of 800 gallons per second; after 15 seconds, the rate is 740 gallons per second, and after 25 seconds, the rate is 700 gallons per second.

(b) Over the interval $0- \leq t \leq 30$, $r(t)$ only has a vertical intercept, representing the initial rate of 800 gallons per second. The horizontal intercept of $t = 200$ is the time (in seconds) when the rate is 0 gallons per second.



(c) Since $r(t)$ is a positive rate of flow over the interval $0 \leq t \leq 30$, the reservoir has the most water over this interval at time $t = 30$ and the least at time $t = 0$.

(d) It is not totally clear whether values of t can sensibly be negative, so the domain could be $-\infty < t < \infty$ or possibly $0 \leq t < \infty$. The range would the be either $-\infty < r(t) < \infty$ or $-\infty < r(t) < 800$.

4. (a) Assuming that the domain is $t \geq 0$, the range of $f(t)$ is $100 \leq f(t) < 2000$.

   (b) $f(0) = 200$, $f(5) \approx 1254.9$ and $f(10) \approx 1963.6$. This means that at the start of the epidemic, 100 people are infected, after 5 days about 1,255 people are infected, and after 10 days about 1,964 people are infected.

5. The domain is $-1 \leq t \leq 4$ and the range is $0 \leq h(t) \leq 9$.

2.3 Piecewise-defined functions

1. (a) The domain is all real numbers, and the range is $-\infty < G(x) < 0$ or $0 \leq G(x) < \infty$.

   (b) The domain of F is all real numbers and the range is $-\infty < F(x) \leq 1$.

2. (a) $g(-2) = -1$, $g(2) = 8$, and $g(0) = 0$.

   (b) The domain is all real numbers and the range is $g(x) = -1$ or $0 \leq g(x) < \infty$.

3. (a) $f(3)$ is undefined

   (b) $f(2) = 2$

   (c) $f(1) = 1$

   (d) $f\left(\frac{1}{2}\right) = \frac{3}{2}$

   (e) $f(0) = 3$

2.4 Preview of transformations: shifts

1. The graph of g is obtained from that of f by a shift 1 unit to the right.

x	-2	-1	0	1	2
$g(x)$	-3	0	2	1	-1

2. The graph of h is obtained from that of f by a shift 1 unit to the left.

x	-1	0	1	2	3
$g(x)$	-3	0	2	1	-1

3. The graph of k is obtained from that of f by a shift 3 units up.

x	-3	-2	-1	0	1
$g(x)$	0	3	5	4	2

4. The graph of m is obtained from that of f by a shift 1 unit to the right and a shift of 3 units up.

x	-1	0	1	2	3
$g(x)$	0	3	5	4	2

5. The domain of $g(x-2)$ is $0 \leq x \leq 9$

6. The range of $R(s) - 150$ is $-50 \leq R(s) - 150 \leq 50$.

7. At age $t = 3$, Jonah's weight is $s(3) + 2$ and at age $t = 6$ it is $s(6) + 2$. In general, Jonah weighs two pounds more than an average weight for a baby his age.

8. (a) $y = g(x) + 2$
   (b) $y = g(x + 2)$

9. $y = f(x + 2) - 3$, where $h = -2$ and $k = -3$.

2.5 Preview of composite and inverse functions

1. (a) $f(g(0)) = f(1) = 2$
   (b) $g(f(0)) = g(3) = -8$
   (c) $g(f(2)) = g(5) = -24$
   (d) $f(g(2)) = f(-3) = -10$
   (e) $f(g(x)) = f(1 - x^2) = 3(1 - x^2) - 1 = 2 - 3x^2$
   (f) $g(f(0)) = g(3) = -8$
   (g) $f(f(x)) = f(3x - 1) = 3(3x - 1) - 1 = 9x - 4$
   (h) $g(g(x)) = g(1 - x^2) = 1 - (1 - x^2)^2 = 1 - (1 - 2x^2 + x^4) = 2x^2 - x^4$

2. $g\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 1 = \frac{9}{4} - 1 = \frac{5}{4}$ and $g^{-1}(-17) = -2$. It can be found as the solution to $2x^3 - 1 = -17$, or $x = -2$.

3. (a) The domain of f^{-1} is the range of f, namely $32 \leq f^{-1}(C) < 127$ and the range is the domain of f, namely $0 \leq C \leq 500$.
   (b) Solve $C = 32 + 0.19m$ for m to get $f^{-1}(C) = \frac{C-32}{0.19}$.

4. The area of the slick is given by $A = \pi r^2$, so the area as a function of time is

$$A = g\left(2t - 0.1t^2\right) = \pi\left(2t - 0.1t^2\right)^2$$

5. (a) $f(10) = 102$
   (b) $f^{-1}(200) = 500$
   (c) $f^{-1}(C) = \frac{C-100)}{0.2}$

6. We can check

$$f(g(x)) = -\frac{2}{-\frac{2}{x+1}} - 1 = (x+1) - 1 = x$$

and

$$g(f(x)) = -\frac{2}{-\frac{2}{x} - 1 + 1} = x$$

so yes, these functions are inverses of each other.

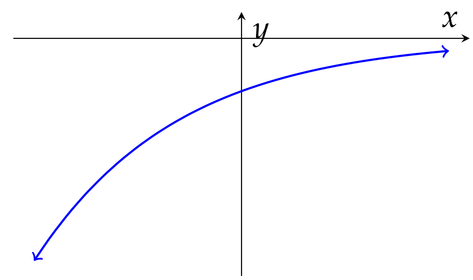
7. (a) $f(0) = 4$
   (b) $f^{-1}(-2) = 4$
   (c) $f^{-1}(0) = 2$
   (d) $f^{-1}(2) = 1$
   (e) $f^{-1}(4) = 0$
   (f) $f^{-1}(3) = \frac{1}{2}$
   (g) $f(f^{-1}(3)) = 3$

8. The formula given in this problem for the surface area of a cube is incorrect. It should be $A = 6s^2$. Answers below are given for both the original problem and, in parentheses, the corrected problem with $A = 6s^2$

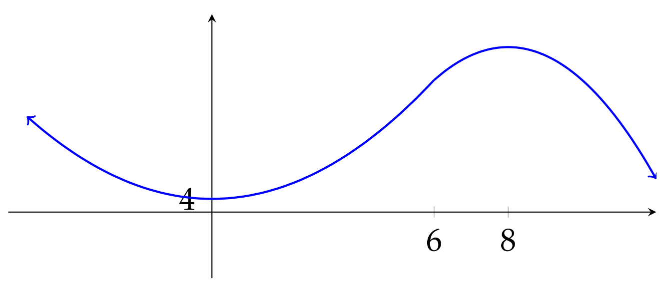
   (a) Since $A = 6s^3$, then $s = f(A) = \sqrt[3]{\frac{A}{6}}$ which would be the side length of a cube with surface area A. (Corrected: Since $A = 6s^2$, then $s = f(A) = \sqrt{\frac{A}{6}}$ which would be the side length of a cube with surface area A.)
   (b) $V = g(f(A)) = \frac{A}{6}$, which would be the volume of a cube with surface area A.
   (Corrected: $V = g(f(A)) = \left(\frac{A}{6}\right)^{\frac{3}{2}}$, which would be the volume of a cube with surface area A.)

2.6 Concavity

1. (a) $\frac{\Delta f}{\Delta x}$ appears to be incresing, so this appears to be a concave up function.
   (b) $\frac{\Delta g}{\Delta t}$ appears to be incresing, so this appears to be a concave up function.
   (c) $y = -x^2$ is concave down.
   (d) $y = x^3$ is concave up for $x > 0$.
2. Here is one.



3. (a) This describes the quantity of the drug in the bloodstream as a function of time. It is a decreasing concave up function.
   (b) The temperature of the hot chocolate as a function of time is also decreasing and cocave up.
4. Here is one.



5. If f is concave down on $0 \le x \le 6$ then the average RoC is smaller over an $3 \le x \le 5$ than over $1 \le x \le 5$. That is,
$$\frac{f(5) - f(3)}{5 - 3} < \frac{f(3) - f(1)}{3 - 1}$$
6. (a) The graph is concave up on the intervals $(-3, -1)$ and $(0, 2)$.
   (b) The graph is concave down on the intervals $(-4, -3)$ and $(-1, 1)$.
   (c) The graph is neither concave up nor concave down on the interval $(2, 4)$.
   (d) The graph is parts concave up, parts concave down on the interval $(-4, 2)$.

3.1 Introduction to the family of quadratic functions

1. (a) $x = 2$ and $x = \frac{3}{2}$
   (b) This factors as $y = (3x + 1)^3$, so there's one zero at $x = -\frac{1}{3}$.

(c) This factors as $N(t) = (t-2)(t-5)$, so there are zeros at $t = 2$ and $t = 5$.

2. (a) $y = \frac{7}{4}(x+2)^2$
   (b) $y = \frac{7}{4}(x-1)(x-4)$

3. $y = \frac{6}{7}(x+1)(x-5)$

4. (a) At time $t = 0$, the velocity is 4 meters per second.
   (b) The object is not moving when $t^2 - 4t + 4 = 0$, or when $t = 2$. The zero can be found by factoring.
   (c) The velicty graph is concave up because the leading term of the quadratic is positive.

5. On the left we have $y = \frac{1}{3}(x+1)(x-3)$ and on the right we have $y = -\frac{5}{12}(x+6)(x-3)$.

3.2 The vertex of a parabola

1. Both the top row are $y = -\frac{1}{9}(x+6)^2 + 9$. In the bottom row, both are $y = \frac{3}{16}(x-6)^2 + 5$.

2. $y = \frac{2}{3}x^2 - 2$.

3. (a) $y = -\frac{3}{8}(x-4)^2 + 2$
   (b) $y = -\frac{2}{49}(x-4)^2 + 2$
   (c) $y = 12\left(x - \frac{1}{2}\right)^2$

6.1 Shifts, reflections, and symmetry

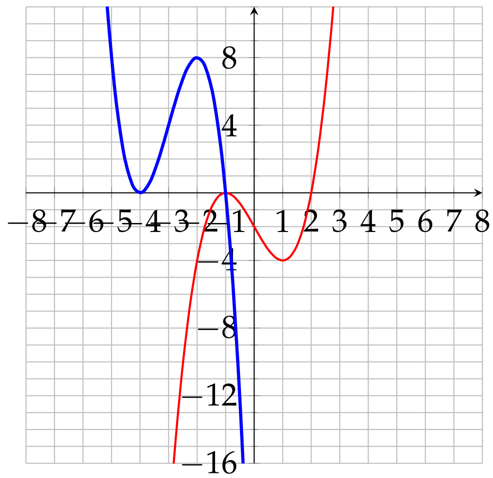
1. (a) $(6, -2)$
   (b) $(6, 2)$

2. (a) Domain $t \leq 0$ and range $-10 < Q(-t) < 1$
   (b) Domain $t \geq 0$ and range $-106 < Q(t) - 6 < -5$
   (c) Domain $t \leq 0$ and range $-1 < -Q(-t) < 10$

3. (a) Since $m(-x) = -m(x)$, m is an odd function.
   (b) Here, ; $p(-x)$ is not the same as $p(x)$ or $-p(x)$, so p is neither even nor odd.
   (c) Since $q(-x) = q(x)$, q is an even function.

6.2 Vertical stretches and compressions

1. Here it is. The transformed graph doesn't actually fit well on the given axes so I've expanded the view a bit.



2. The graph of the new function is obtained from that of r by a vertical compression by a factor of $\frac{1}{3}$, a reflection about the vertical axis, and a horizontal compression by a factor of $\frac{1}{2}$.

3. (a) $P(t-20)$
   (b) $P(t)+8$
   (c) $3P(t)$
   (d) $P(t)-1$.

Horizontal stretches and combinations of transformations

1. $y=-f(2x)+2$

2. The transformations are

   (a) A vertical stretch by a factor of 4.
   (b) A shift down by 5 units.
   (c) A horizontal compression by a factor of $\frac{1}{3}$.

   Item 1 and 2 must be in that order, but item 3 can be done anywhere in the list.

3. (a) $(16,-4)$
   (b) $(8,-2)$
   (c) $(-16,-4)$
   (d) $(4,4)$

11.1 Power functions and proportionality

1. Since $\frac{e^{2x}}{4x^{13}} \to \infty$ as $x \to \infty$ (see this graphically or numerically) e^{2x} dominates $4x^{13}$.

2. (a) The constant of proportionality is $k=3$.
   (b) $c=3t^2$
   (c) $c=48$ when $t=-4$.

11.2 Polynomial functions

1. (a) Degree 4, leading term $3x^4$; $y \to \infty$ as $x \to \pm\infty$.
   (b) Degree 5, leading term $-x^5$; $y \to \infty$ as $x \to -\infty$ and $y \to -\infty$ as $x \to \infty$.
   (c) Degree 2, leading term $-2x^2$; $y \to -\infty$ as $x \to \pm\infty$.

2. (a) $\lim_{x\to\infty} (x^2 - x) = \infty$
   (b) $\lim_{x\to-\infty} (1 - x - 4x^3) = \infty$
   (c) $\lim_{x\to\infty} \left(\frac{1}{5}x^4 - 2x^3 + 5\right) = \infty$

11.3 The short-run behavior of polynomials

1. Going left-to-right:
   (a) $y = 4(x+3)(x+1)$
   (b) $y = -\frac{3}{2}(x+4)(x+2)(x-2)$

2. Assuming there are no other zeros, this polynomial could be $f(x) = x$ or $f(x) = x^2$.

3. (a) The x intercepts are $1, -2$, and ± 4. The y intercept is 32.
   (b) The x intercepts are $0, 6$ and -1. The y intercept is 0.
   (c) The x intercepts are $-4, 3$ and 4. The y intercept is 48.

11.4 Rational functions

1. (a) $\lim_{n\to\infty} \frac{3n^2}{n^2+5} = 3$
   (b) $\lim_{x\to-\infty} \frac{1}{(x-2)(x+1} = 0$
   (c) $\lim_{t\to\infty} \frac{t^3+2}{t-7} = \infty$

2. (a) Over a long time, the oxygen level approaches 1 again.
   (b) Solve $\frac{t^2-t+1}{t^2+1} = \frac{3}{4}$ to get $t = 2 \pm \sqrt{3}$. (You'll need to use the quadratic formula.) Checking the graph, you can see that the larger intercept is the one where the value returns to 75% of its original level; this is $t = 2 + \sqrt{3} \approx 2.7$ weeks.

3. (a) p has a horizontal asymptote at $y = 0$.
   (b) w has a horizontal asymptote at $y = -\frac{3}{2}$.

11.5 The short run behavior of rational functions

1. (a) The y intercept is $-\frac{3}{8}$.
   (b) The function has a zero at $x = -3$.

(c) The function has vertical asymptotes at $x = 4$ and $x = -2$.

2. $y = \dfrac{x-2}{(x+1)}$