Math 118 - Fall 2024 - Common Final Exam, version A Solutions

1. (9 points) The revenue of Madeline's tutoring business is \$10,000 in the year 2024. Recall that a linear function has a general form of $P = mt + b$ and an exponential function has a general form of $P = a \cdot b^t$.

   (a) If the revenue of Madeline's business is decreasing by \$300 per year, find a formula for the function $R(t)$, the revenue t years after 2024.

   Solution: $R(t) = -300t + 10,000$

1 pt	some linear function
1 pt	correct slope
1 pt	correct intercept
3 points	

   (b) If Madeline's revenue is instead increasing by 12% per year, find a formula for the function $R(t)$, the revenue t years after 2024.

   Solution: $R(t) = 10\,000 \cdot 1.12^t$

1 pt	some exponential function
1 pt	correct growth factor
1 pt	correct initial value
3 points	

   (c) Using your formula from part b, find the year that Madeline's revenue will hit 30,000. Round to the nearest whole number.

   Solution:

$$\begin{aligned} 10\,000 \cdot 1.12^t &= 30\,000 \\ 1.12^t &= 3 \\ t\ln(1.12) &= \ln(3) \\ t &= \frac{\ln(3)}{\ln(1.12)} \\ &\approx 10 \text{ years} \end{aligned}$$

1 pt	set up correct equation based on part (b)
1 pt	progress on solution
1 pt	correct solution
3 points	

2. (5 points) Find the half-life of a radioactive substance that decays at a continuous rate of 7% per hour.

Solution:

$$
\begin{aligned}
e^{-0.07t} &= \tfrac{1}{2} \\
-0.07t- &= \ln\left(\tfrac{1}{2}\right) \\
t &= \frac{\ln\left(\frac{1}{2}\right)}{-0.07} \\
&\approx 9.902 \text{ hours}
\end{aligned}
$$

2 pt	set up correct equation for half life
2 pt	progress on solution
1 pt	correct solution
5 points	

Students may also earn the first 4 points if they remember and correctly use a formula for half life. In this case, the last point is for correctly evaluating their formula.

3. (10 points) Determine if the following functions are linear, exponential, or neither. Then, find an equation for each function or explain why it cannot be done.

(a)

x	-1	0	1	2	3
$f(x)$	-7	-5	-3	-1	1

Solution: This is a linear function: $f(x) = -5 + 2x$.

1 pt	correctly identify linear function, or write a linear formula. 2 pt
correct slope 2 pt	correct intercept
5 points	

(b)

x	0	1	2	3	4
$g(x)$	3	6	12	24	48

Solution: This is a linear function: $f(x) = 3 \cdot 2^x$.

1 pt	correctly identify exponential function, or write an exponential formula.
2 pt	correct growth factor
2 pt	correct initial value
5 points	

4. (11 points) Elliot opens a bank account with an initial deposit of \$20,000. It earns interest at a nominal rate of 3% per year.

(a) Find the balance of their account after 5 years if interest is compounded as follows.

i. Annually (once per year).

Solution:

$$B = 2\,000 \cdot 1.03^5 \approx 23\,185.48$$

1 pt	evidence of correct exponential growth formula
1 pt	correctly plug in values from the problem
1 pt	correct evaluation
3 points	

ii. Quarterly (4 times per year).

Solution:

$$B = 2\,000\left(1 + \frac{0.03}{4}\right)^{4c\cdot 5} \approx 23\,223.68,$$

1 pt	evidence of correct compounded interest formula
1 pt	correctly plug in values from the problem
1 pt	correct evaluation
3 points	

iii. Continuously.

Solution:

$$B = 2\,000e^{0.03\cdot 5} \approx 23\,236.68$$

1 pt	evidence of correct compounded interest formula
1 pt	correctly plug in values from the problem
1 pt	correct evaluation
3 points	

(b) Of the three accounts listed in part (a) of this question, which has the highest effective rate? That is, which account increases the most over the course of one year?

Solution: The last one, with interest compounded continuously, has the highest effective rate.

2 points for correct answer (no reason is required).

5. (8 points) Consider the exponential function $Q = 182(0.97)^t$.

(a) Determine if this function displays exponential growth or decay.

Circle one: **exponential growth** or **exponential decay**. Explain your answer in a sentence.

> **Solution:** The correct choice is "exponential decay". This is because the factor 0.97 is less than 1, and so it is a decay factor.
> 1 point for an answer which gives a correct reason.

(b) Give the initial value for this function.

> **Solution:** 182 (1 point)

(c) Give the growth rate for this function. Write your answer as a percentage.

> **Solution:**
> $$0.97 - 1 = -0.03 = -3\%$$

1 pt	answer may be correct, or may just give the decay *factor*
1 pt	answer correctly give growth rate as a negative percentage
2 points	

(d) Give the domain for the given function.

> **Solution:** All (real) numbers. (1 point)

(e) Give the range for the given function.

> **Solution:** All positive numbers. (1 pt)

(f) Write the given function in the form $Q = ae^{kt}$.

> **Solution:** Solve $e^k = 0.97$ to get $k = \ln(0.97) \approx -0.0305$. Then $Q \approx 182e^{-0.0305t}$.

1 pt	at least one of a and k appear correctly
1 pt	correct formula given height2 points

> A correct answer could give a reasonably rounded value of k or use the exact expression $\ln(0.97)$.

6. (11 points) The height of a spring oscillates sinusoidally between 3 feet at $t = 0$ seconds and 11 feet at $t = 5$ seconds. Consider the function, $H = f(t)$, which gives the height, H, as a function of time, t, in seconds.

(a) Find the amplitude, period, and midline of the function $H = f(t)$.

- The amplitude is **4 feet**
- The period is **10 sec.**
- The midline is **7 feet**

Solution: 0.5 points each for a total of 1.5 points. Units are not required.

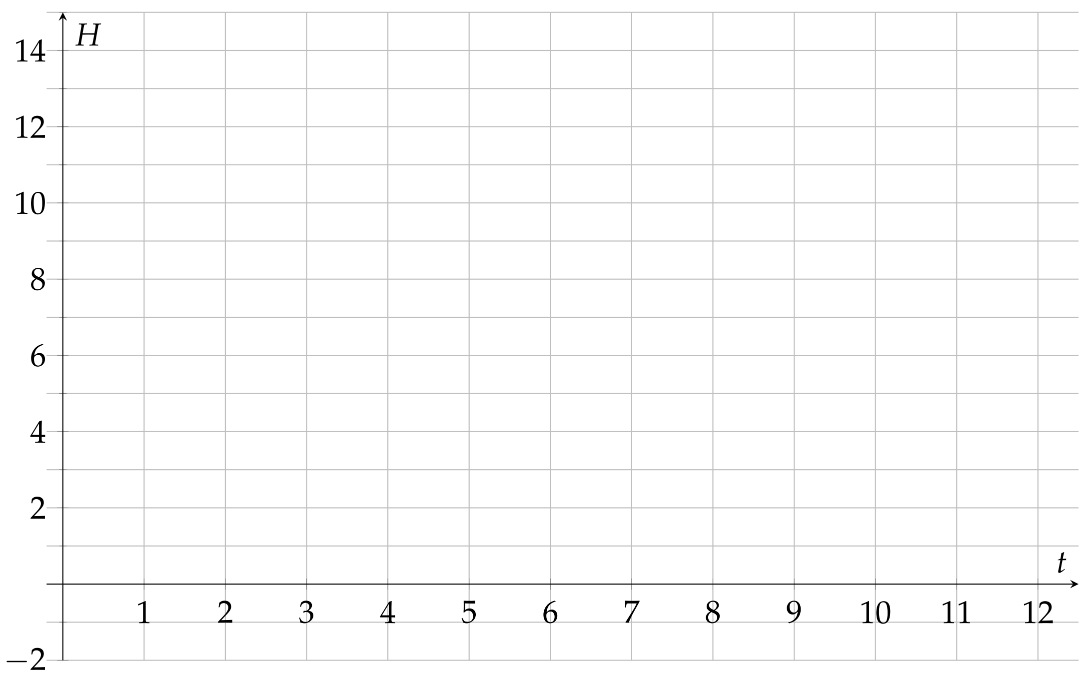
(b) Find a formula for the population, $H = f(t)$.

Solution:

$$H(t) = -4\cos\left(\frac{\pi}{5}t\right) + 7$$

0.5 pt	correct amplitude
0.5 pt	correct average value
1 pt	correct function choice and horizontal shift value
0.5	correct B term computed from period
2.5 points	

(c) Graph H as a function of t over the whole interval shown below.



Solution:

0.5 pt	correct sinusoidal shape
0.5 pt	correct amplitude
0.5 pt	correct average value
0.5	correct period
0.5	graph covers the whole interval
2.5 points	

In the case where part (a) gives incorrect amplitude, period, or midline, it is recommended to grade elements (b) and (c) as correct if they match *either* the description in the problem or the corresponding values in part (a).

(d) Write an equation whose solutions give all times that the height of the spring is 6 feet.

Solution:

$$-4\cos\left(\frac{\pi}{5}t\right)+7=6$$

0.5 pt	some equation is given
1 pt	formula from (b) equal to 6
1.5 points	

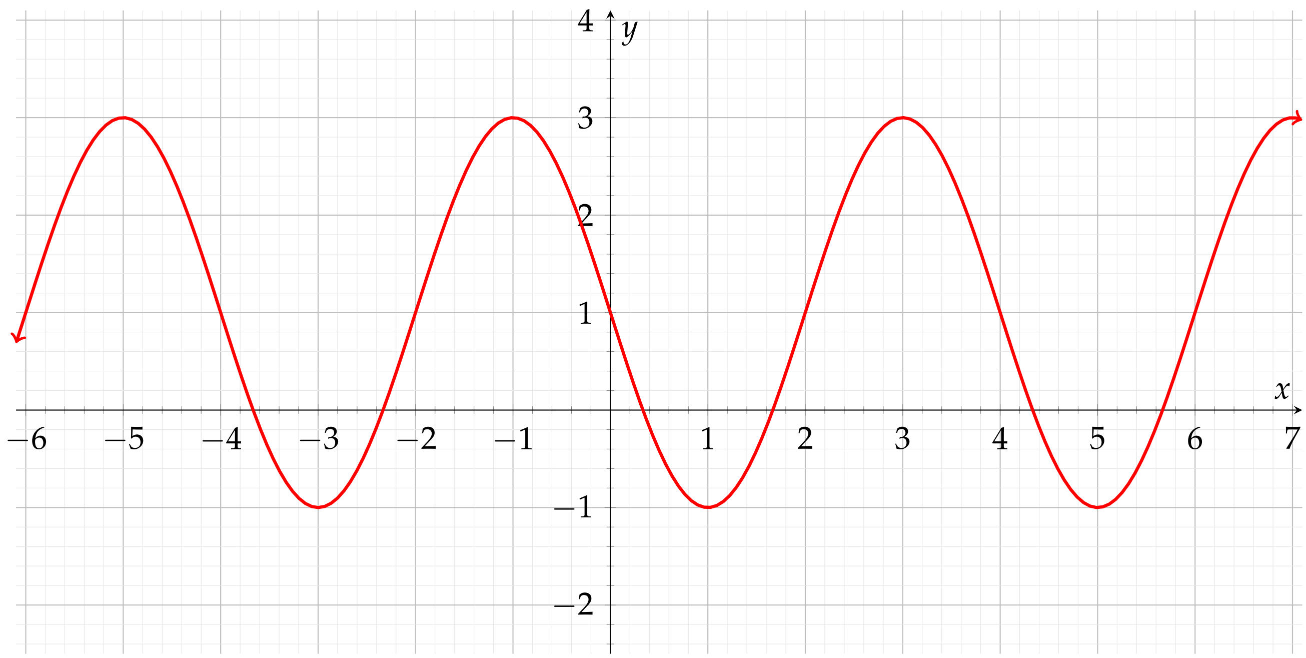
(e) Find a solution to the equation you found in part (d), giving your answer in terms of an inverse trig function and also evaluate with correct units.

Solution:

$$\begin{aligned}\cos\left(\frac{\pi}{5}t\right) &= \frac{1}{4}\\ \frac{\pi}{5}t &= \cos^{-1}\left(\frac{1}{4}\right)\\ t &= \frac{5}{\pi}\cos^{-1}\left(\frac{1}{4}\right)\\ &\approx 2.10 \text{ seconds}\end{aligned}$$

1 pt	use inverse cosine
1 pt	correctly solve for t
1 pt	correct units height3 points

7. (6 points) Find a formula of the trigonometric function shown in the graph below.



Solution:

$$H(t) = -2\sin\left(\frac{\pi}{2}x\right) + 1$$

1 pt	correct amplitude
1 pt	correct average value
2 pt	correct function choice and horizontal shift value
1 pt	evidence of correctly identifying the period
1	correct B term computed from period
6 points	

8. (8 points) For an angle α where $\pi < \alpha < \frac{3\pi}{2}$ such that $\sin(\alpha) = -\frac{3}{4}$, find the given quantities without finding α. Give an exact answer for each part.

(a) $\cos(\alpha)$

Solution:

$$\cos^2(\alpha) + \left(-\tfrac{3}{4}\right)^2 = 1$$
$$\cos^2(\alpha) = \tfrac{7}{16}$$
$$\cos(\alpha) = -\tfrac{\sqrt{7}}{4}$$

1 pt	use Pythagorean identity (or theorem with an appropriate triangle)
1 pt	progress on algebra to solve for $\cos(\alpha)$
1 pt	correct solution and sign choice height3 points

Rationalizing the denominator is not required.

(b) $\tan(\alpha)$

Solution:

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{-3/4}{-\sqrt{7}/4} = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$$

1 pt	use correct definition for tangent
1 pt	computations correct
2 points	

Rationalizing the denominator is not required.

(c) $\cos(\alpha - \frac{\pi}{4})$

Solution:

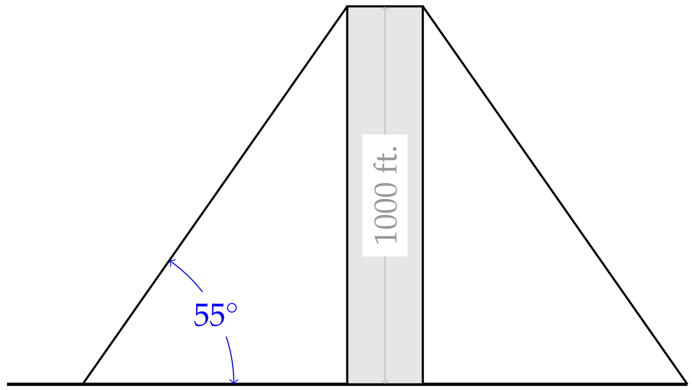
$$\begin{aligned} \cos(\alpha - \frac{\pi}{4}) &= \cos(\alpha)\cos\left(\tfrac{\pi}{4}\right) + \sin(\alpha)\sin\left(\tfrac{\pi}{4}\right) \\ &= -\tfrac{\sqrt{7}}{4} \cdot \tfrac{\sqrt{2}}{2} - \tfrac{3}{4} \cdot \tfrac{\sqrt{2}}{2} \\ &= \frac{-\sqrt{14} - 3\sqrt{2}}{8} \end{aligned}$$

1 pt use correct cosine angle sum formula

1 pt correctly plug in exact values for $\cos(\alpha)$ and $\sin(\alpha)$

1 pt correctly plug in exact values for $\cos(\pi/4)$ and $\sin(\pi/4)$

9. (6 points) The top of a 1000 foot tower is to be anchored by cables that make an angle of 55° with the ground. A diagram of the situation is below.



(a) How long must the cables be?

Solution: With the length of the cable as x:

$$\sin(55°) = \frac{1000}{x}$$
$$x = \frac{1000}{\sin(55°)}$$
$$\approx 1\,220.77 \text{ feet}$$

1 pt	correct trig function
1 pt	correct equation
1 pt	correct solution height3 points

(b) How far from the base of the tower should the anchors be placed?

Solution: With the distance along the ground as d:

$$\tan(55°) = \frac{1000}{d}$$
$$d = \frac{1000}{\tan(55°)}$$
$$\approx 700.21 \text{ feet}$$

1 pt	correct trig function
1 pt	correct equation
1 pt	correct solution height3 points

This part could also be solved with the Pythagorean theorem, 1 point each for using it, setting it up correctly, and solving for d.

10. (5 points) Let $f(x) = 2x - 5$, $g(x) = -4x + 3$ and $h(x) = \cos(x)$. Find the following, and simplify your answers completely:

(a) $f(g(2))$

Solution:

$$\begin{aligned} f\big(g(2)\big) &= f\big(-4(2)+3\big) \\ &= f(-5) \\ &= 2(-5)-5 \\ &= -15 \end{aligned}$$

1 pt	at least one function evaluated correctly
1 pt	correct solution
2 points	

(b) $h(g(f(x)))$

Solution:

$$\begin{aligned} h\Big(g\big(f(x)\big)\Big) &= h\big(g(2x-5)\big) \\ &= h\big(-4(2x-5)+3\big) \\ &= h(-8x+23) \\ &= \cos(-8x+23) \end{aligned}$$

1 pt	at least one function evaluated correctly
1 pt	at least two functions evaluated correctly
1 pt	correct solution
3 points	

11. (6 points) Let $P = f(t) = 700(1.06)^t$ be the population of a town. Let t be measured in years since 2024.

(a) Evaluate $f(8)$. Round to the nearest whole number. Describe in words what this quantity represents. Write your answer in a complete sentence with units.

Solution:

$$f(8) = 700 \cdot 1.06^8 \approx 1\,116.$$

8 years after 2024 (or in 2032) the town's population will be 1,116 people.

1 pt	correctly evaluate $f(8)$
1 pt	sentence correctly gives a population after 8 years
2 points	

(b) Find a formula for $f^{-1}(P)$ in terms of P. Give an exact answer.

Solution:

$$\begin{aligned} P &= 700 \cdot 1.06^t \\ 1.06^t &= \frac{P}{700} \\ t\ln(1.06) &= \ln\left(\frac{P}{700}\right) \\ t &= \frac{\ln\left(\frac{P}{700}\right)}{\ln(1.06)} \end{aligned}$$

So $f^{-1}(P) = \frac{\ln\left(\frac{P}{700}\right)}{\ln(1.06)}$.

1 pt	correctly set up $P = f(t)$ to solve
1 pt	correct solution for $f^{-1}(P)$ height2 points

An answer which gives the inverse expression as "$t =$" rather than "$f^{-1}(P) =$" is OK.

(c) Evaluate $f^{-1}(1000)$. Round to the nearest whole number

Solution: $f^{-1}(1000) = \frac{\ln\left(\frac{1000}{700}\right)}{\ln(1.06)} \approx 6$

1 pt for plugging into the formula from the previous part

(d) Describe in words what the quantity you found in part (c) represents. Write your answer in a complete sentence with units.

Solution: It will take 6 years for the population to reach 1,000 people.

1 pt	solution from previous part is years
1 pt	1000 is population
1 pt	correct sentence and units
3 points	

12. (5 points) Decompose the function

$$f(x) = \frac{4}{\ln(3x+6)}$$

into a composition of two new functions u and v, where v is the inside function. That is $f(x) = u(v(x))$, so that $u(x) \neq x$ and $v(x) \neq x$.

Solution: Of course, there are infinitely many correct solutions. Here's one:

$$v(x) = \ln(3x+6) \text{ and } u(x) = \frac{4}{x}$$

2 pt	at least one function which could be composed to get f
2 pt	two functions which could be composed to get f
1 pt	correct order of composition $u(v(x))$ is given
5 points	

13. (5 points) Perform the following conversions.

Solution: 1 point overall for this problem for exact answers without deimal approximations.

(a) Convert the Cartesian coordinates $(x, y) = (5, 5)$ to polar coordinates. Give an exact answer.

Solution: $\tan(\theta) = 1$ and $r^2 = 5^2 + 5^2$, so $(r, \theta) = \left(5\sqrt{2}, \frac{\pi}{4}\right)$.

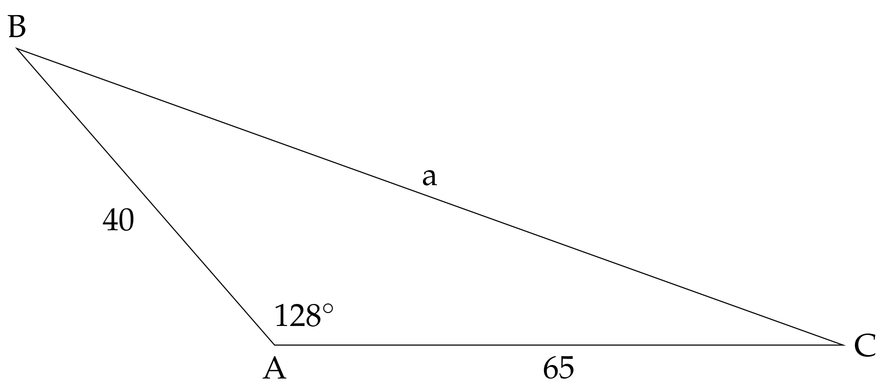
1 pt	evidence of correct conversion formulas
1 pt	correct computations
2 points	

(b) Convert the polar coordinates $(r, \theta) = (7, \dfrac{2\pi}{3})$ to Cartesian coordinates. Give an exact answer.

Solution: $x = 7\cos\left(\frac{2\pi}{3}\right)$ and $y = 7\sin\left(\frac{2\pi}{3}\right)$, so $(x, y) = \left(-\frac{7}{2}, \frac{7\sqrt{3}}{2}\right)$.

1 pt	evidence of correct conversion formulas
1 pt	correct computations
2 points	

14. (5 points) Find the length of the missing side, a, in the diagram below.



Solution: Use the Law of Cosines to write

$$a^2 = 40^2 + 65^2 - 2 \cdot 40 \cdot 65\cos(128°),$$

or $a \approx 95.01$

1 pt	use Law of cosines
2 pt	plug into LoC correctly
1 pt	progress on evaluation, may have order of operations error
1 pt	corret evaluation height5 points