Math 161 - 2022 Spring - Common Final Exam

Name: SOLNS

Section Number: _______ Instructor Name: ________________

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- This exam has 12 questions worth a total of 150 points. Please check that your exam is complete, but otherwise do not look at the exam until the official start.

- Fill in your name and section above.

- Show your work. Correct work without corresponding work may not receive credit.

- You have 120 minutes to complete this exam.

- Technology of any kind is prohibited. The use of any notes is prohibited.
1. (20 points) Compute $\frac{dy}{dx}$ for:

(a) $y = (\tan x)e^{1x}$

$$y' = (\tan x)'e^{1x} + \tan x(e^{1x})'$$

$$= \sec^2 x \cdot e^{1x} + \tan x(1 \cdot e^{1x})$$

(b) $y = (x^3 - 1)^3 = u^3$

$$y' = 3u^2 u'$$

$$= 3(x^3 - 1)^2 (3x^2 - 1)'$$

$$= 3(x^3 - 1)^2 (9x^2)$$

(c) $y = \cos(\sin(\pi x)) = \cos(u)$

$$y' = -\sin(u)u'$$

$$= -\sin(\sin(\pi x))(\sin(\pi x))'$$

$$= -\sin(\sin(\pi x)) \cos(\pi x) \pi$$

(d) $y = \frac{11x + 13 - \frac{17}{x^2}}{x^{1/3}}$. Then compute $\frac{dy}{dx}$.

$$y = 11x^{-1/3} + 13x^{-2/3} - 17x^{-5/3}$$

$$y' = 11\left(-\frac{1}{3}x^{-4/3}\right) + 13\left(-\frac{2}{3}x^{-5/3}\right) - 17\left(-\frac{5}{3}x^{-8/3}\right)$$

$$= -\frac{11}{3}x^{-4/3} - \frac{26}{3}x^{-5/3} + \frac{85}{3}x^{-8/3}$$

$$y'' = -\frac{11}{3}\left(-\frac{4}{3}x^{-7/3}\right) + \left(-\frac{26}{3}\right)\left(-\frac{5}{3}x^{-8/3}\right) + \frac{85}{3}\left(-\frac{8}{3}x^{-11/3}\right)$$
2. (8 points) The graph of the function $f(x)$ is shown below:

Based on the graph of $f$, answer the following:

(a) $\lim_{x \to -0.5} f(x) = -1$

(b) $\lim_{x \to 1} f(x) = -\infty$

(c) $\lim_{x \to 0.5} f(x) = 0$

(d) $f(-0.5) = 1.2$

3. (10 points) Consider

$$y = 3x^5 - 20x^3 - 75x + 999.$$ 

Find all critical points and all inflection points. You do not have to classify the critical points, but you do have to distinguish between potential inflection points and actual inflection points.
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\[ y' = 15x^4 - 60x^2 - 75 \]
\[ = 15(x^4 - 4x^2 - 5) \]
\[ = 15(x^2 - 5)(x^2 + 1) \]
\[ = 15(x + \sqrt{5})(x - \sqrt{5})(x^2 + 1) \]

\[ \text{C.P. @ } x = \pm \sqrt{5} \]

\[ y'' = 60x(x + \sqrt{5})(x - \sqrt{5}) \]

Potential I.P. @ \( x = 0, \pm \sqrt{5} \)

All potential inf. pts are inf. pts. as \( y'' \) changes sign across \( 0, -5, 5 \)

4. (10 points) Consider

\[ y = \frac{x^2 + 6x + 9}{2x^2 - 18} \]

Find all horizontal and vertical asymptotes. Classify any other discontinuities that exist.

\[ \text{H.A.} \]
\[ \lim_{x \to \infty} y = \lim_{x \to \infty} \frac{1 + \frac{6}{x} + \frac{9}{x^2}}{2 - \frac{18}{x^2}} = \frac{1}{2} \]

\[ \text{V.A.} \]
Set denom = 0:
\[ 2x^2 - 18 = 0 \]
\[ x^2 - 9 = 0 \]
\[ x = \pm 3 \]

But
\[ y = \frac{(x+3)^2}{2(x+3)(x-3)} \]
so \( x = -3 \) is a remov. disc. and \( x = 3 \) is only V.A.

5. (15 points) The graph of the equation \( 3x^2 = 2y^2 + 8xy + 1 \) is a hyperbola as shown below:
(a) The hyperbola intersects the x-axis twice. Find the x values of those two points. (Eyeing this is not good, as they are not rational numbers.)

\[ \text{Set } y = 0 \]
\[ 3x^2 = 1 \]
\[ x = \pm \sqrt{\frac{1}{3}} \]

(b) Implicit differentiation yields \( 6x = 4yy' + 8(y + xy') \). Solve for \( y' \).

\[ 6x = 4yy' + 8y + 8xy' \]
\[ 6x - 8y = y'(4y + 8x) \]
\[ y' = \frac{6x - 8y}{4y + 8x} \]

(c) Find the x values at which the tangent line is vertical.

\[ \text{Set } \frac{dy}{dx} = \text{something} \text{ over } 0 \text{ or } \Delta x = 0 : \quad 4y + 8x = 0 \]

when \( x = -\frac{y}{2} \)

Plug \( x = \pm \sqrt{\frac{1}{3}} \) into original fn:

\[ 3x^2 = 2y^2 + 8xy + 1 \]
\[ 3 \left( \pm \sqrt{\frac{1}{3}} \right)^2 = 2y^2 + 8 \left( \pm \sqrt{\frac{1}{3}} \right) \left( \pm \sqrt{\frac{1}{3}} \right) + 1 \]
\[ 3 \cdot \frac{1}{3} = 2y^2 + 8 \cdot \frac{1}{3} + 1 \]
\[ 11x^2 = 1 \quad \text{or} \quad x = \pm \sqrt{\frac{1}{11}} \]

6. (10 points) Find the point on the line \( y = 3x - 2 \) which is closest to the origin.

\[ d^2 = (x - x_1)^2 + (y - y_1)^2 \]
\[ d^2 = (x - 0)^2 + (y - 0)^2 \]
\[ 12 \quad \text{or} \quad 12 \cdot 12 \quad \text{or} \quad 12 \]

\[ \Rightarrow x = \frac{12}{20} = \frac{3}{5} \]

Then \( y = 3 \left( \frac{3}{5} \right) - 2 \)
\[ = -\frac{1}{5} \]
7. (10 points) A streetlight is mounted at the top of a 15 foot pole. A 6 foot tall person walks away from the streetlight at 2 feet per second. How fast is the length of the person’s shadow growing when the person is 23 feet from the pole?

Setup:
Similar D’s:
\[ \frac{u}{6} = \frac{u + x}{15}, \quad x = 23 \]

Units: \( u \) and \( x \) are in feet
\( u’ \) and \( x’ \) are \( \frac{du}{dt}, \frac{dx}{dt} \), so they are in ft/sec

\[ u' = \frac{2(2)}{3} = \frac{4}{3} \text{ ft/sec} \]

8. (15 points) Consider the function \( y = f(x) = e^{3x} \).

(a) Find the equation of the line \( L(x) \) which is tangent to \( f(x) \) when \( x = 2 \).

\[ f(2) = e^6 \]
\[ f'(x) = 3e^{3x} \]

\[ \text{Plugging into} \quad y - y_1 = m(x - x_1) \]
\[ y - e^6 = 3e^6(x - 2) \]
\[ f(x) = 3e^{3x} \]
\[ f'(x) \big|_{x=2} = 3e^6 \]

\[ y - e^6 = 3e^6(x-2) \]

\[ L(x) = f(c) + f'(c)(x-c) \]
\[ = e^6 + 3e^6(x-2) \]

(b) Find the third-order Taylor polynomial centered at \( c = 2 \) for \( f(x) \).

\[ P_3(x) = f(2) + f'(2)(x-2) + f''(2)(x-2)^2 + f'''(2)(x-2)^3 \]

| \( k \) | \( f^{(k)}(2) \) | \( f^{(k)}(2) \big|_{x=2} \) | \( k! \) | \( ak \) |
|---|---|---|---|---|
| 0 | \( e^{3x} \) | \( e^6 \) | 1 | \( e^6 \) |
| 1 | \( 3e^{3x} \) | \( 3e^6 \) | 1 | \( 3e^6 \) |
| 2 | \( 9e^{3x} \) | \( 9e^6 \) | 2 | \( \frac{9}{2}e^6 \) |
| 3 | \( 27e^{3x} \) | \( 27e^6 \) | 6 | \( \frac{27}{6}e^6 \) |

\[ P_3(x) = e^6 + (3e^6)(x-2) + \left( \frac{9}{2}e^6 \right)(x-2)^2 + \left( \frac{27}{6}e^6 \right)(x-2)^3 \]

9. (15 points) Evaluate the following limits

\( \lim_{x \to 3} \frac{x^2 - 4}{x^2 - 5x + 6} \)  
\[ \left[ \frac{DE}{\text{L'Hop}} \right] \quad \frac{y-4}{y-10+6} = \frac{0}{0} \]

\[ \lim_{x \to 2} \frac{2x}{2x-5} \]  
\[ \left[ \frac{DE}{\text{L'Hop}} \right] \quad \frac{4}{4} = \frac{4}{1} = -4 \]

\( \text{OR} \) algebra: simplify to \( \frac{(x-2)(x-2)}{(x-2)(x-3)} = \)
OR algebra: simplify to \( \frac{(x+2)(x-2)}{(x-2)(x-3)} = \frac{x+2}{x-3} = \frac{y}{-1} = -y \)

So \( \lim_{x \to 2} \frac{x+2}{x-3} = \frac{4}{-1} = -4 \)

(b) \( \lim_{x \to 1^+} \frac{e^x - e}{\ln x} \)

\[ \text{DE} \quad \frac{e^1 - e}{\ln 1} = \frac{0}{0} \quad \text{L'Hôpital} \]

\[ \lim_{x \to 1^+} \frac{e^x}{\ln x} = \lim_{x \to 1^+} xe^x \]

\[ \text{DE} \quad \frac{e^1}{1} = e \]
10. (10 points) Evaluate the following indefinite integrals:

(a) \[ \int x + 1 + \frac{1}{x} + \frac{1}{x^2} \, dx \]
\[ = \frac{x^2}{2} + x + \ln |x| + \frac{x^{-1}}{-1} + C \]

(b) \[ \int \frac{3x}{\sqrt{2 + x^2}} \, dx \]
\[ u = 2 + x^2 \]
\[ \frac{du}{dx} = 2x \]
\[ \frac{du}{2x} = dx \]
\[ \int \frac{3x}{\sqrt{u}} \left( \frac{du}{2x} \right) \]
\[ = \frac{3}{2} \int u^{-1/2} \, du \]
\[ = \frac{3}{2} \left( 2u^{1/2} \right) + C \]
\[ = 3(2 + x^2)^{1/2} + C \]
11. (15 points) Evaluate the following definite integrals:

(a) \( \int_{0}^{5} \sqrt{25 - x^2} \, dx \)

Geometry: Upper half circle of radius 5 from [0, 5]

So, \( \int_{0}^{5} \sqrt{25 - x^2} \, dx = \frac{A}{4} = \frac{\pi (5)^2}{4} = \frac{25\pi}{4} \)

(b) \( \int_{\pi/6}^{\pi/2} 1 + \sin(x) \, dx \)

\[ = \left[ x - \cos x \right]_{\pi/6}^{\pi/2} \]

\[ = \left( \frac{\pi}{2} - \cos \frac{\pi}{2} \right) - \left( \frac{\pi}{6} - \cos \frac{\pi}{6} \right) \]

\[ = \left( \frac{\pi}{2} - \frac{\pi}{6} \right) - \left( \cos \frac{\pi}{2} - \cos \frac{\pi}{6} \right) \]

\[ = \frac{\pi}{3} - \left( 0 - \frac{\sqrt{3}}{2} \right) \]

\[ = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \]

12. (12 points) Multiple choice

(a) Which function is an antiderivative of \( f(x) = \ln(x) \)?
12. (12 points) Multiple choice

(a) Which function is an antiderivative of \( f(x) = \ln(x) \)?

(I) \( \frac{1}{x} \)  
(II) \( x \ln x - x \)  
(III) \( x \ln x + x \)  
(IV) \( \frac{(\ln x)^2}{2} \)

(b) \( \int_1^7 (e^{x^2} \cos(x))(x^5 - x^3) \, dx = \)

(I) 0  
(II) \( (e^{t^2} \cos(t))(t^5 - t^3) \)  
(III) \( (e^{t^2} \cos(t))(t^5 - t^3) + C \)

For the next two parts, the graph of the function \( f \) is shown below:

(c) Using the graph above, which is the largest quantity?

(I) \( \int_1^3 f(x) \, dx \)  
(II) \( \int_3^5 f(x) \, dx \)  
(III) \( \int_1^5 f(x) \, dx \)

(d) Using the graph above, from \( x = 2 \) to \( x = 4 \), which is the largest quantity?

(I) \( L_2 \)  
(II) \( R_2 \)  
(III) \( T_2 \)

\text{would never be largest or smallest}