Math 131 – Answer Key and grading guidelines

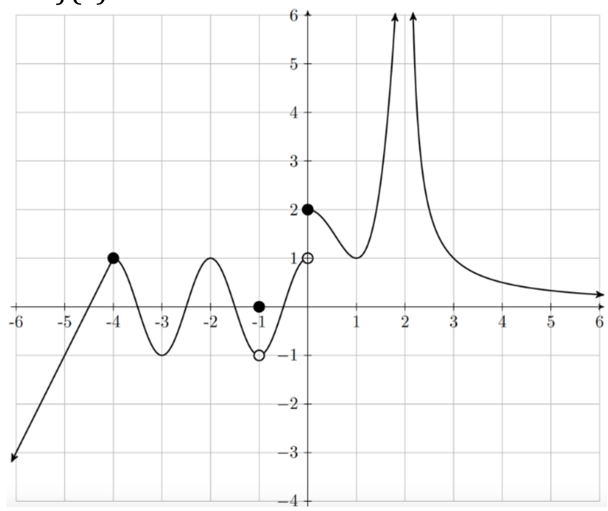
1. (8 points – 2 points each part) Use the graph of the function $f(x)$, to find the following values

   (a) $\lim_{x\to -1} f(x) = -1$ (2 pts)

   (b) $\lim_{x\to 0+} f(x) = 2$ (2 pts)

   (c) $\lim_{x\to -4} f(x) = 1$ (2 pts)

   (d) Identify the intervals where $f(x)$ is continuous.




$f(x)$ is continuous on the following intervals

$$(-\infty, -1) \cup (-1, 0) \cup (0, 2) \cup (2, \infty)$$

Partial credit for each interval: 0.5 pts.
It's okay if students replace $-\infty$ by -6 and $\infty$ by 6.

2. (4 points) Find the limit

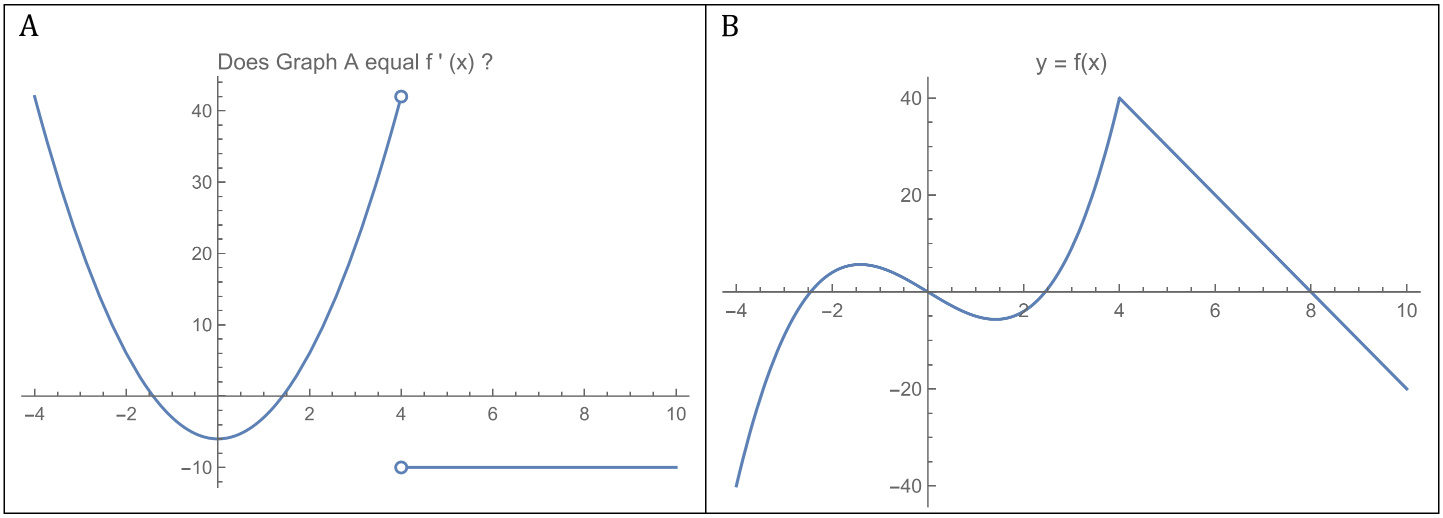
$$\lim_{x\to\infty} \frac{6x^3 - 5x + 7}{3 + x - 3x^2 - 2x^3}$$

$$= \lim_{x\to\infty} \frac{6 - \frac{5}{x^2} + \frac{7}{x^3}}{\frac{3}{x^3} + \frac{1}{x^2} - \frac{3}{x} - 2} \quad \text{(2 pts)}$$

$= \frac{6}{-2} = -3$ (2 pts)

Or, any appropriate explanation (2 pts) and result (2 pts).

3. (6 points) Could the first graph, A be the derivative of the second graph, B? Explain your answer, making use of these words: first derivative, second derivative, critical point, inflection point as appropriate and giving ***at least four criteria*** that lead you to your conclusion.




Answer: Yes. (2 points)
4 points: One point for each of at least 4 correct observations – samples are below:

- f(x) has 3 critical points, A has two zeros and one undefined point
- f(x) is not differentiable at x = 4, A is undefined at x = 4
- f(x) has critical points at x = -1.75, +1.75 and A has zeros at -1.75, +1.75
- f(x) is decreasing with constant slope for x >4. A has the constant value -10
- f is decreasing for x in the intervals (-1.75, 1.75) and x > 4. A is negative for those x
- f is increasing for x in the intervals (-Infinity, -1.75), ( 1.75, 4). A is positive for those x
- f(x) has an inflection point at x = 0, A has a local min at x = 0.

4. (9 points – 3 points each part) The depth, h (in mm), of the water runoff down a slope during a steady rain is a function of the distance, x (in meters), from the top of the slope. So $h = f(x)$.
   (a) What are the units of $f'(15)$? mm per m (3 points)
   (b) What does $f'(15) = 0.03$ mean in practical terms? (Include units, and the meaning of both 15 and 0.03 in your answer.
   At a distance of 15 meters from the top of the slope, the depth of water runoff is increasing by 0.03 mm per additional meter of distance from the top.
   1 point: interpret 15 with units
   1 point: correct units for 0.03
   1 point: correct interpretation
   (c) Given that $f(15) = 4$ and $f'(15) = 0.03$, estimate the depth of the water runoff 17 meters from the top of the slope.
   3 points
   $f(17) \approx f(15) + f'(15)(17 - 15) = 4.06\, mm$

5. Partial credit at instructor's discretion

Find the derivative of $f(x) = \ln\,[sin^3(7x-2)]$. Show your work.

(6 points)

**Answer:** $\frac{\mathbf{21\ cos\,(7x-2)}}{\boldsymbol{sin}\mathbf{(7x-2)}}$

6. Find a formula for the slope of the tangent line to $y = 3(x-9)^2$ when $x = b$

(5 points)

**Answer:** $\boldsymbol{slope = \ y'(b) = 6(b-9)}$

7. Partial credit at instructor's discretion

Find the derivative of $g(x) = \sqrt{3e^x + e^{x^5}}$

(6 points)

**Answer:** $\frac{3e^x + 5x^4 e^{x^5}}{2\sqrt{3e^x + e^{x^5}}}$, simplification, fraction format not required

8. Four variants which I am assuming are to be solved without a calculator.

$$f(x) = x - 2ln(x\ +\ 1), where\ 0\ \le\ x\ \le\ 2$$

(8 points) Find the global maximum and minimum for the function on the closed interval

ANSWERS

$$f(x) = x - 2ln(x\ +\ 1), where\ 0\ \le\ x\ \le\ 2$$

**Phase 1:** 2 points for correctly finding the first derivative
1 point for trying to find the first derivative

$$f\,'(x) = 1 - \frac{2}{x+1}$$

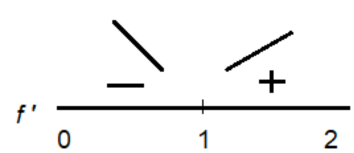
**Phase 2:** 2 points for correctly finding the critical point using the first derivative.
1 point for setting the first derivative equal to zero.

$0 = 1 - \frac{2}{x+1} \rightarrow \frac{2}{x+1} = 1 \rightarrow 2 = x + 1 \rightarrow 1 = x$

**Phase 3:** (2 points)

a) One possibility: First derivative test. Sign analysis using the first derivative.

1 point to attempt it 2 points to do it correctly.




$f\,'(0) = 1 - \frac{2}{0+1} = 1 - \frac{2}{1} = 1 - 2 = -1$ Therefore the slope is negative to the left of 1 and the function f is decreasing to the left of x = 1.

$f\,'(2) = 1 - \frac{2}{2+1} = 1 - \frac{2}{3} = \frac{1}{3}$ Therefore the slope is positive to the right of 1 and the function f is increasing to the right of x = 1.

Conclusion: f(1) or x =1 must be a global minimum.

b) Another possibility: The second derivative test. Sign analysis using the second derivative. 1 point to attempt it and 2 points to do it correctly.

$$f''(x) = 2(x+1)^{-2} \rightarrow f''(x) = \frac{2}{(x+1)^2} \rightarrow$$

The second derivative is always positive on the interval 0 to 2. Thus f is always concave up on the interval in which the critical point x =1 exists. So the critical point must be a global minimum.

**Phase 4:** Finding the global maximum. 2 points if done correctly, 1 point if attempted.

Plug the endpoints into the original function. (The student may, but does not have to, plug the critical point into the original function.)
$f(0) = 0 - 2ln(0 + 1) = 0 - 2\ln(1) = 0 - 2(0) = 0$ Global maximum.
$f(1) = 1 - 2ln(1 + 1) = 1 - 2\ln(2) = -0.38629$
$f(2) = 2 - 2ln(2 + 1) = 2 - 2\ln(3) = -0.19722$

Without a calculator, it is possible to recognize that $ln(3)$ is bigger than $ln(e)$ and that $ln(e) = 1$ so $ln(3)$ is slightly larger than 1. Thus $2ln(3)$ is slightly greater than 2 and $2 - 2\ln(3)$ is negative. A negative is less than zero, so x = 0 is the global maximum.

9. (8 points) The revenue for selling $q$ items is $R(q) = 600 \cdot q - 2 \cdot q^2$ and the total cost of producing $q$ items is $C(q) = 120 + 60 \cdot q$. What quantity maximizes profit? Find the value of the profit function at this production level.

Short Answer: q = 135, Maximum profit = $36,330

ANSWERS: Rubric:
Phase 1: Maximizing profit
Setting MR = MC Maximum profit occurs when MR = MC
*(1 pt.* for finding MR*) MR = R'*$(q)$ = 600 −4·q
(1 pt. for finding MC) MC = $C'(q) = 60$.
(1 pt. for setting them equal to each other) $600 - 4q = 60$
(1 pt. for correctly solving for q) $540 = 4q \rightarrow 135 = q$
OR
Take the derivative of the profit function, set it equal to zero.
(1 pt for finding the profit equation) $\pi(q) = R(q) - C(q) \rightarrow \pi(q) = 600q - 2q^2 - (120 + 60q) \rightarrow \pi(q) = 600q - 2q^2 - 120 - 60q$

$$\pi(q) = 540q - 2q^2 - 120$$

(1 pt. for taking its derivative) $\pi'(q) = 540 - 4q \rightarrow$
(1 pt. for setting it equal to zero) $0 = 540 - 4q \rightarrow$
(1 pt for finding q) $4q = 540 \rightarrow q = 135$

Phase 2: Value of the profit function when q = 135

(2 pts for finding the profit equation)

$$\pi(q) = R(q) - C(q) \to \pi(q) = 600q - 2q^2 - (120 + 60q) \to \pi(q) = 600q - 2q^2 - 120 - 60q$$

$$\pi(q) = 540q - 2q^2 - 120$$

(1 pt. for plugging 135 in for q) $\pi(135) = 540(135) - 2(135)^2 - 120$

(1 pt. for finding the profit) $\pi(135) = 540(135) - 2(135)^2 - 120 = 36330$

The value of the profit function when q is 135 is $36,330.

10. (9 points – 3 points each) Find the following limits, use l'Hopital's Rule if applicable:

(a) $\lim_{x\to 0}\left(\frac{e^{x^2}-1}{3\cos(x)-3}\right) = -\frac{2}{3}$

(b) $\lim_{x\to\infty}\left(\frac{e^{x}}{\cos(x)-1}\right) = DNE$ , $-\infty$

(c) $\lim_{x\to\infty}\frac{\ln(x)+x^2}{3x+1} = \infty$

11. ( 7 points) Use the table to estimate $\int_0^{40} f(x)\,dx$ with n=5 (Average left-hand and right-hand sums)

| $x$ | 0 | 8 | 16 | 24 | 32 | 40 |
|---|---|---|---|---|---|---|
| $f(x)$ | 72 | 68 | 55 | 47 | 39 | 28 |

Solution for Pb 11- V1

- (1 point) From the values of $x$ determine the length of each interval $\Delta x = 8$
- (1 points) Write the left- hand sum $LH = \sum_{i=0}^{4} f(0 + i\,\Delta x)\times\Delta x = (72+68+55+47+39)\times 8$
- (1 point) Calculate the left- hand sum $LH = 2,248$
- (1point) Write the right-hand sum $RH = \sum_{i=1}^{5} f(0 + i\,\Delta x)\times\Delta x = (68+55+47+39+28)\times 8$
- (1 point) Calculate the right-hand sum $RH = 1,896$

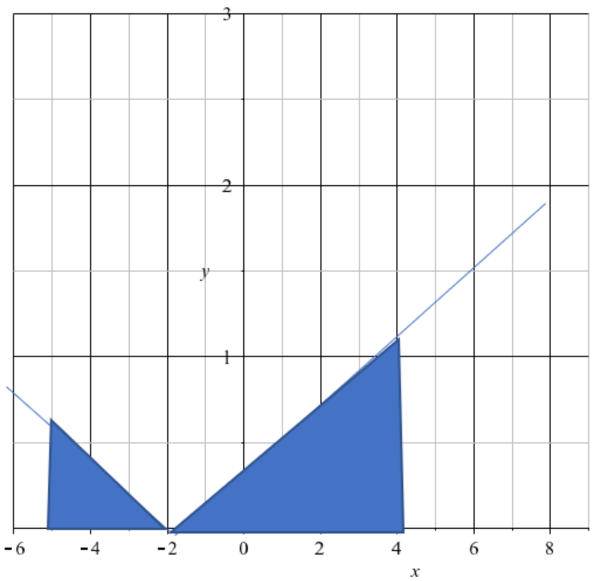
(2 points) Estimate $\int_0^{40} f(x)\,dx \simeq \frac{LH+RH}{2} = \frac{2248+1896}{2} = 2072$

12. (7 points) Find

Find $\int_{-5}^{4} \frac{|x+2|}{5}\,dx$ geometrically. *(Sketch the graph.)*

Solution for Pb 12- V 1

- (3 points) Sketch the graph of the integrant $\frac{|x+2|}{5}$ , and mark the area that is $\int_{-5}^{4} \frac{|x+2|}{5} dx$




- (4 points) Find the area of the two triangles

$$\int_{-5}^{4} \frac{|x+2|}{5} dx = \frac{1}{2} f(-5) \times (-2-(-5)) + \frac{1}{2} f(4)(4-(-2))$$

$$= \frac{1}{2} \times \frac{3}{5} \times 3 + \frac{1}{2} \times \frac{6}{5} \times 6$$

$$= \frac{45}{10}$$

$$= 4.5$$

13. (6 points) The total emissions are $\int_0^6 4\sqrt{t} + 8 \, dt \approx 87.19$ kilograms of carbon dioxide.
- 3 points: set up definite integral correctly
- 2 points: evaluate definite integral correctly
- 1 point: correct units

14. (5 points) $\int 3x^4 - 4^x + \cos(x) \, dx = \frac{3}{5}x^5 - \frac{4^x}{\ln 4} + \sin(x) + C$
- 4 points: antiderivatives correct
- 1 point: constant of integration included

15. (6 points) Find the definite integral, leave exact answers:

a) $\int_1^8 3\sqrt{x} \, dx = 32\sqrt{2} - 2$

b) $\int_0^{\frac{\pi}{3}} (5\sin(t) + 4) dt = \frac{5}{2} + \frac{4\pi}{3}$

For each part: 2 points for a correct antiderivative, full 3 points for correctly evaluating the integral. If there are multiple reasonable versions of the exact answer, it is up to instructors to decide what they would require from their students.