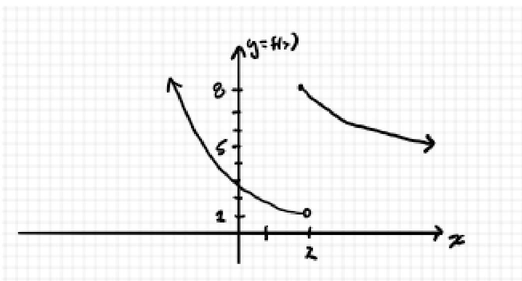
Math 131 Sample Common Final Exam Solutions

Note: Original questions are shown in black, and solutions are written in blue.

Chapter 1. End Behavior, Limits, and Continuity

1. Sketch the graph of one function that has all of these properties:
   (a) $\lim_{x\to\infty} f(x) = 5$
   (b) $\lim_{x\to 2^+} f(x) = 8$
   (c) $\lim_{x\to 2^-} f(x) = 1$
   (d) $f(x)$ decreases for $x > 2$
   (e) Evaluate the limit $\lim_{x->\infty} f'(x)$ for the function you graphed.

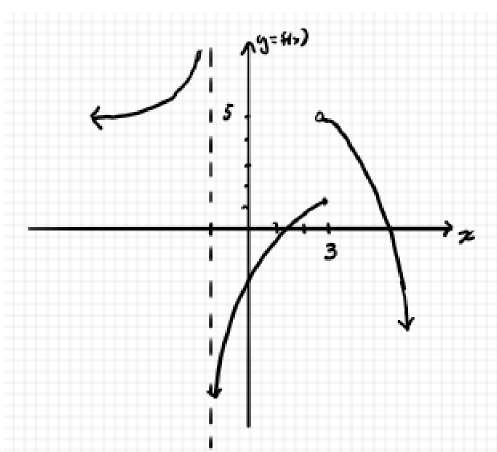
Here is one solution:



For part (e): $\lim_{x\to\infty} f'(x) = 0$

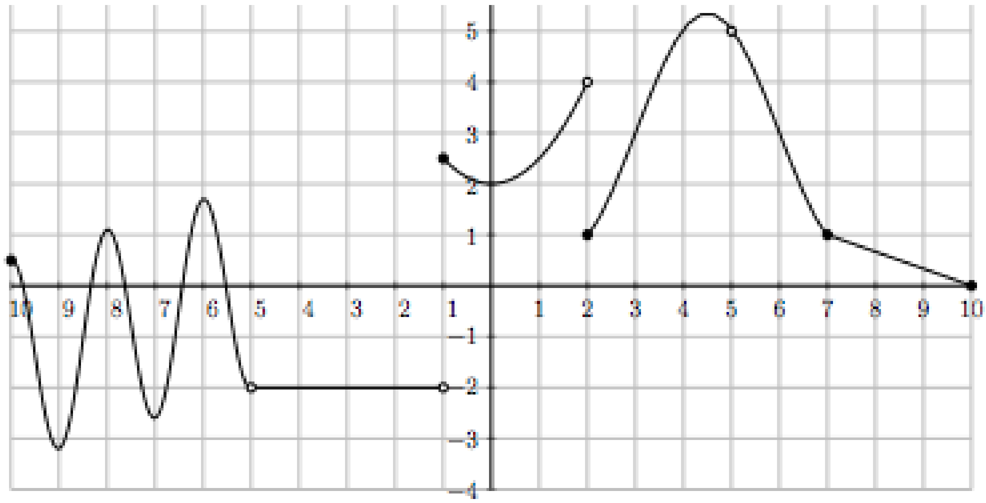
2. Sketch the graph of one function that has all of these properties:
   (a) $f(x)$ is continuous everywhere EXCEPT at $x = 3$ and at $x = -1$[1]
   (b) $\lim_{x\to\infty} f(x) = -\infty$
   (c) $\lim_{x\to 3^+} f(x) = 5$
   (d) $\lim_{x\to -\infty} f(x) = 5$
   (e) $f(x)$ has a vertical asymptote at $x = -1$.

Here is one solution:



[1]This portion in red was meant to be included in the original question, but was left out by mistake.

3. Use the graph of the function $f(x)$ to find the following values:



(a) $\lim_{x\to 0} f(x) =$ 2

(b) $\lim_{x\to -1^+} f(x) =$ 2.5

(c) $\lim_{x\to -1^-} f(x) =$ -2

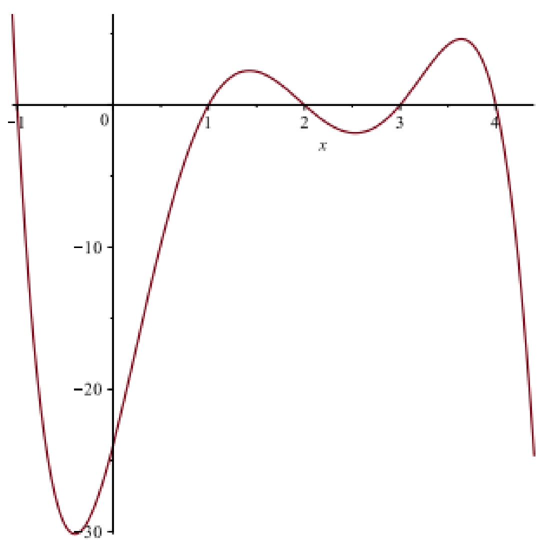
(d) $\lim_{x\to -1} f(x)$ does not exist

(e) $\lim_{x\to 5} f(x) =$ 5

(f) $f(5)$ is undefined

(g) f is continuous over $[-10,-5)$, $(-5,-1)$, $(-1,2)$, $(2,5)$, and $(5,10]$.

4. What is the minimum possible degree of the polynomial function shown below? Is the leading coefficient positive or negative?



The degree is an odd number at least 5, and its leading coefficient is negative.

5. Which function dominates in the long run (as $x \to \infty$), $y = 70e^{0.2x}$ or $y = 2000x^6$?

$y = 70e^{0.2x}$ dominates $y = 2000x^6$ in the long run.

6. Find the limit

$$\lim_{x\to\infty} \frac{6x^3 - 5x + 7}{3 + x - 3x^2 - 2x^3} = \boxed{-3}.$$

Chapter 2. Definition and Meaning of the Derivative

1. Here is some recorded data for two functions, f and g

x	1	2	5	8	9	10
$f(x)$	1	2	4	16	8	4
$g(x)$	15	10	6	8	13	15

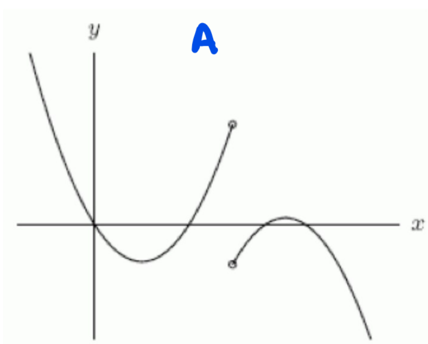
(a) f is not the derivative of g and g is not the derivative of f. Give evidence to support these claims.

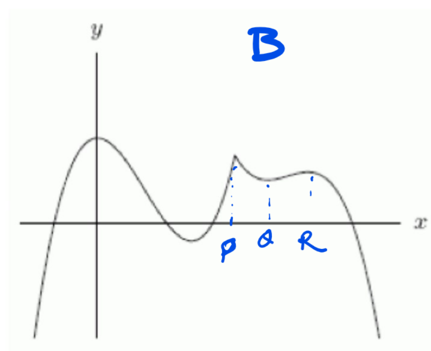
- g is not the derivative of f because f appears to be decreasing over $8 < x < 10$, but g is positive on this interval.
- f is not the derivative of g because on the same interval $1 < x < 8$, g appears to be decreasing but f is positive.

(b) Use a difference quotient to estimate $g'(10)$.

$g'(10) \approx 2$ using the difference quotient over $9 \le x \le 10$.

2. Could the first graph, A be the derivative of the second graph, B? Explain your answer.





There are several features that match up appropriately. Here are two:

- At the origin, graph B has a horizontal tangent line and graph A has a value of zero.
- At point P, graph B has an abrupt change in slope from positive on the left to negative on the right. Graph A has a matching abrupt change in value from positive to negative.

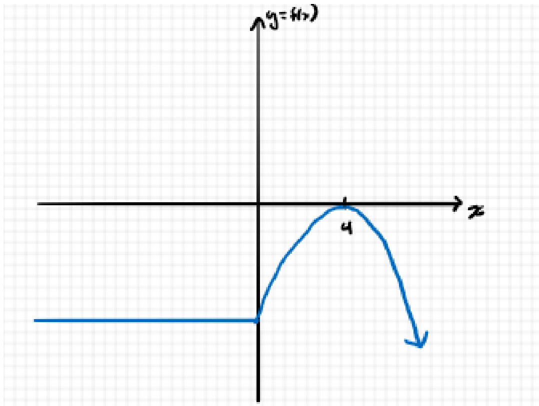
3. Draw the graph of a continuous function $y = g(x)$ that satisfies the following three conditions.

(a) $g'(x) = 0$ for $x < 0$

(b) $g'(x) > 0$ for $0 < x < 4$

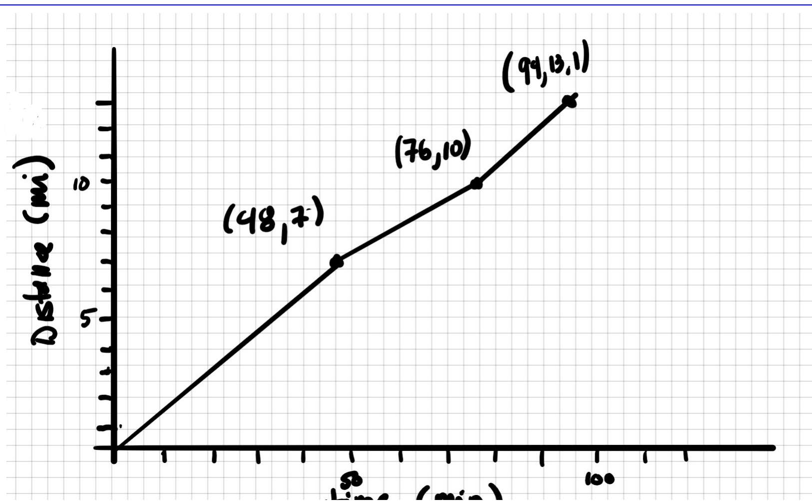
(c) $g'(x) < 0$ for $x > 4$

Here is one solution:



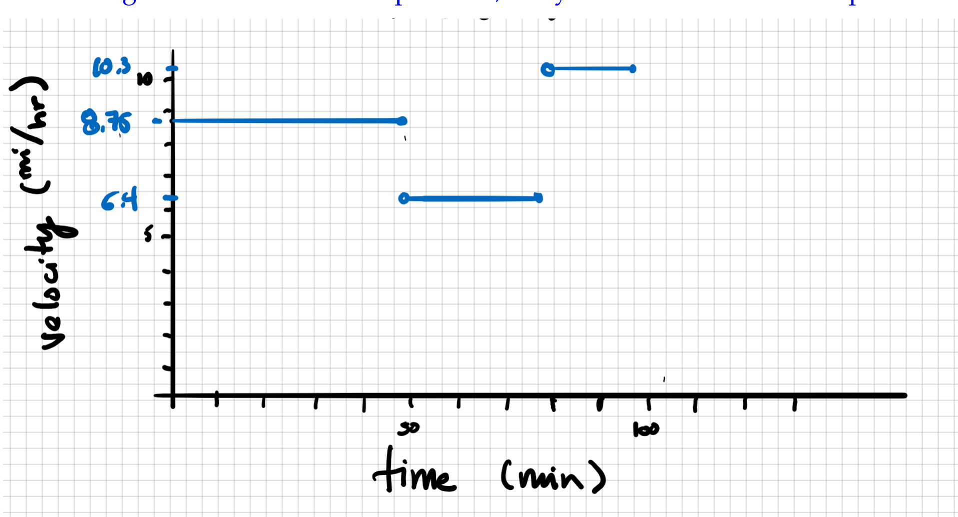
4. A runner competed in a half marathon in Anaheim, a distance of 13.1 miles. She ran the first 7 miles at a steady pace in 48 minutes, the second 3 miles at a steady pace in 28 minutes and the last 3.1 miles at a steady pace in 18 minutes.

   (a) Sketch a well-labeled graph of her distance completed with respect to time.



   (b) Sketch a well-labeled graph of her velocity with respect to time.

I've chosen to give velocities in miles per hour, but you could also use miles per minute.



5. On the axes below, sketch a smooth, continuous curve (i.e., no sharp corners, no breaks) which passes through the

point $P(3,2)$, and which clearly satisfies the following conditions:

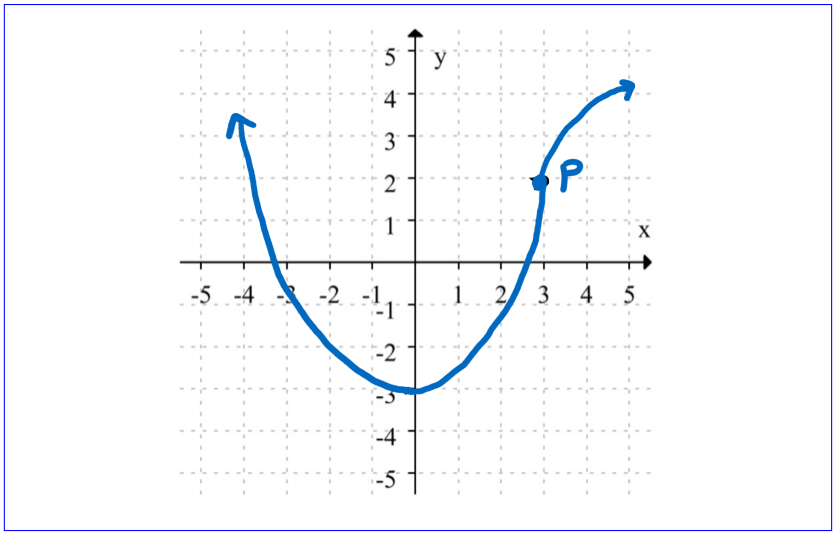
(a) Concave up to the left of P.

(b) Concave down to the right of P.

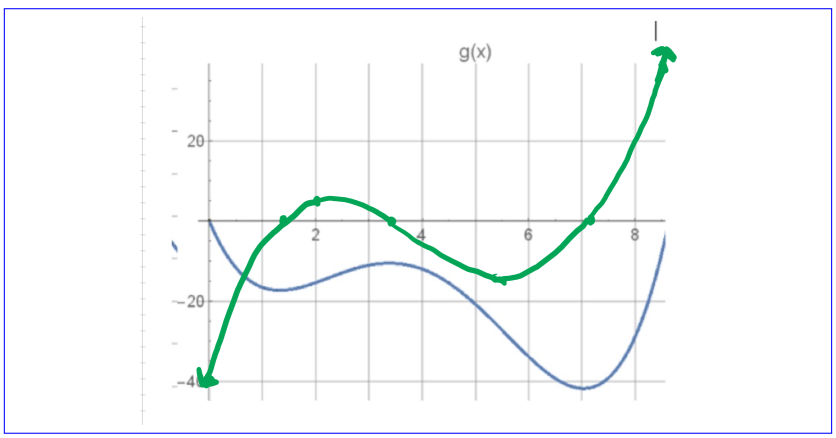
(c) Increasing for $x > 0$.

(d) Decreasing for $x < 0$.

(e) Does not pass through the origin.



6. Sketch a graph of $g'(x)$ given this graph[2] of $g(x)$.



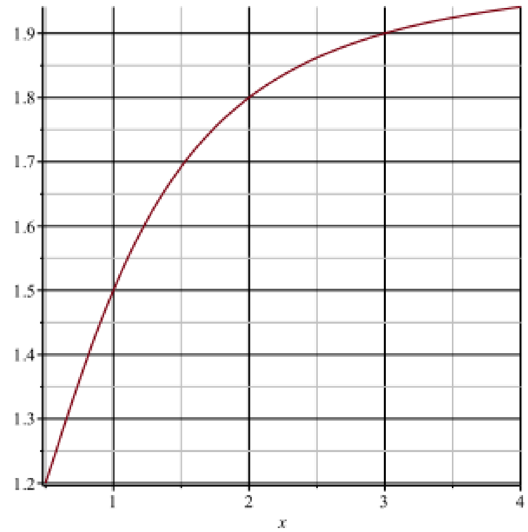
7. The graph of $p(t)$, in the following figure, gives the position of a particle p at time t. List the following quantities in order, from smallest to largest.

(a) average velocity on $1 < t < 3$.

(b) instantaneous velocity at $t = 1$.

(c) instantaneous velocity at $t = 3$.

[2]Note that $g(x)$ is shown in blue and $g'(x)$ is shown in green.



In order from smallest to largest:

- instantaneous velocity at $t = 3$
- average velocity over $1 < t < 3$
- instantaneous velocity at $t = 1$

8. The air in the troposphere (the lowest layer of the Earths atmosphere) is getting colder as you move up vertically above the Earth. Let $t(h)$ be the temperature in degrees Celsius at a height h (in meters) above the surface of the Earth.

   (a) What is the sign of $t'(h)$?

   (b) What are the units of $t'(1200)$?

   (c) What does $t'(1200)$ mean in practical terms?

(a) $t'(h)$ is negative

(b) $t'(1200)$ has units of degrees Celsius per meter

(c) $t'(1200)$ is the instantaneous rate of change of air temperature with respect to height when at 1200 meters above the earth.

9. An economist is interested in how the price of a certain item affects its sales. At a price of \$$p$, a quantity, q, of the item is sold. If $q = f(p)$,

   (a) Is $f(p) = q$ positive or negative? Give a reason for your answer?

   (b) What is the practical meaning of $f'(150) = -25$, including units?

(a) For any price p, $f(p)$ should be positive, because it represents a quantity of some product sold.

(b) $f'(150) = -25$ means that at a price of \$150, sales are decreasing with respect to price at an instantaneous rate of 25 items per dollar of price.

Chapter 3. Procedural Derivative Calculations

1. Find the derivative of $f(x) = \ln[\sin^3(7x)]$. Show all of your work.

$$f'(x) = \frac{21sin^2(7x)\cos(7x)}{\sin^3(7x)}$$

2. Find a formula for the slope of the tangent line to $y = (x-9)^2$ when $x = b$.

When $x = b$, the slope of the tangent line is $2b - 18$.

3. Consider the graph $y = e^x$. What formula would you use to find the x-intercept of the tangent line to the graph at (a, e^a)?

We would use the formula for the tangent line

$$y - y(a) = y'(a)(x-a),$$

and rearrange to slope-intercept form. The x intercept is $a - 1$.

4. Determine the derivative rule for finding the derivative of the reciprocal function $\frac{1}{g(x)}$.

(a) $\frac{-g'(x))}{[g(x)]^2}$ (b) $\frac{1}{g'(x)}$ (c) $[g(x)]^{-1}$ (d) $\frac{g'(x)}{2g(x)}$

Use A, $\frac{-g'(x))}{[g(x)]^2}$

5. What is the instantaneous rate of change of the function $f(x) = e^{-x^2}$ at $x = 2$? Round to 3 decimal places.

$f'(2) \approx -0.073.$

6. Find the derivative of $g(x) = \sqrt{e^x + e^{x^5}}$

(a) $\sqrt{e^x + 5x^4e^{x^5}}$ (b) $\frac{1}{2\sqrt{e^x + e^{x^5}}}$ (c) $\frac{e^x + 5x^4e^{x^5}}{2\sqrt{e^x + e^{x^5}}}$ (d) $\frac{e^x + x^5e^{x^5-1}}{2\sqrt{e^x + e^{x^5}}}$

C, $g'(x) = \frac{e^x + 5x^4e^{x^2}}{2\sqrt{e^x + e^{x^2}}}$.

7. Your friends have been working through some practice problems for the final. They have asked for your help in finding tangent line to the graph of the function $h(x) = \ln[\cos(2x)]$ at $x = \pi$. Their steps are listed below. Identify the errors in these steps and help them to correct their mistakes.

$$h'(x) = \frac{\sin(2x)}{2\cos(2x)}$$
$$h(\pi) = \ln[\cos(2\pi)] = \ln(0) = -1$$
$$y = -1 + x \cdot \frac{\sin(2x)}{2\cos(2x)}$$

They have circled the last line, indicating that

$$y = -1 + x \cdot \frac{\sin(2x)}{2\cos(2x)}$$

is the equation of the tangent line at $x = \pi$.

First step errors The answer should be negative, and the factor of 2 should be in the numerator: $h'(x) = \frac{-2\sin(2x)}{\cos(2x)}$.

Second step errors $\cos(2\pi) = 1$ and $\ln(0)$ is undefined. In fact, $h(\pi) = \ln(\cos(2\pi)) = \ln(1) = 0$

Third step errors They forgot to plug π into $h'(x)$ to find the slope when $x = \pi$. The result is not a linear function! The correct equation of the tangent line is $y = 0$.

8. Your friend has been working through some practice problems for the final. They have asked for your help in finding the tangent line for $f(x) = \cos^2(-x^2)$ at $x = 0$. Their steps are listed below. Identify the errors in these steps and help them correct their mistake, include suggestions for your friend to help them navigate problems similar to this one.

$$f'(x) = 2(-2x)\cos(-x^2)$$
$$f(0) = \cos^2(0) = 1$$
$$y = 1 - 4x\cos(-x^2)$$

They have circled the last line, indicating that $y = 1 - 4x\cos(-x^2)$ is the equation for the tangent line at $x = 0$.

First step errors This is not a correct use of the chain rule: they have missed the derivative of the cosine function. It should be $f'(x) = 2\cos(-x^2)(-\sin(-x^2))(-2x)$. Since there are three functions composed (square, cosine, and $-x^2$), there should be three functions in the "chain" of derivatives.

Second step errors This is correct.

Third step errors They forgot to evaluate the derivative at $x = 0$ to find the slope. The result is not a linear function! As it happens, even their incorrect derivative would have given the correct slope of zero at $x = 0$. The equation of the tangent line at $x = 0$ is $y = 1$.

Chapter 4. Optimization, Applications of the Derivative

1. Find the critical points of the function

$$h(x) = x + \frac{4}{x}$$

and classify them as either local maxima or local minima.

There are two critical points: $x = 2$ gives a local minimum and $x = -2$ gives a local maximum.

2. Consider the following function, where a is a nonzero constant.

$$f(x) = \frac{a}{x^2} + x$$

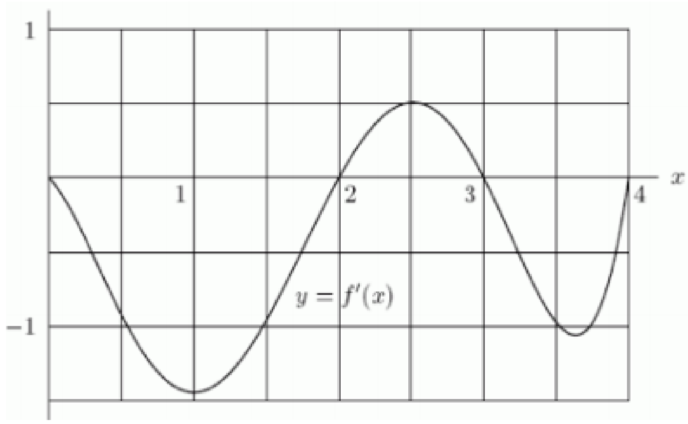
(a) Find the critical point(s) of f(x).

There is a single critical point at $x = \sqrt[3]{2a}$.

(b) Use the second derivative test to determine whether the function has a local maximum or local minimum.

- $f''(\sqrt[3]{2a}) = \frac{6a}{x^4} > 0$ if $a > 0$, in which case the critical point is a local minimum.
- $f''(\sqrt[3]{2a}) = \frac{6a}{x^4} < 0$ if $a < 0$, in which case the critical point is a local maximum.

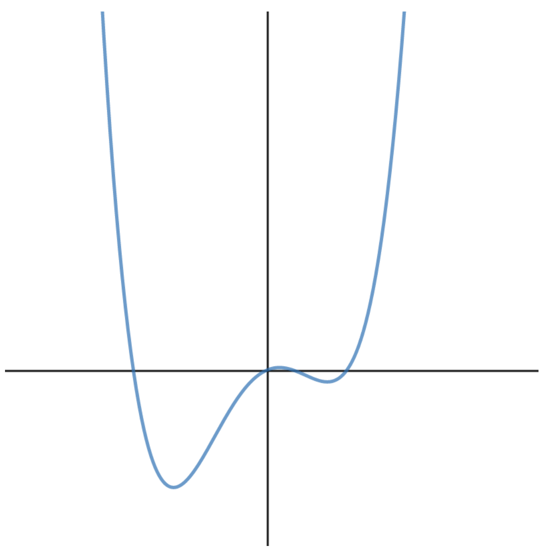
3. Below is the graph of the derivative of a function f, i.e., it is a graph of $y = f'(x)$. Where in the interval $0 \leq x \leq 4$ does f achieve its global maximum?



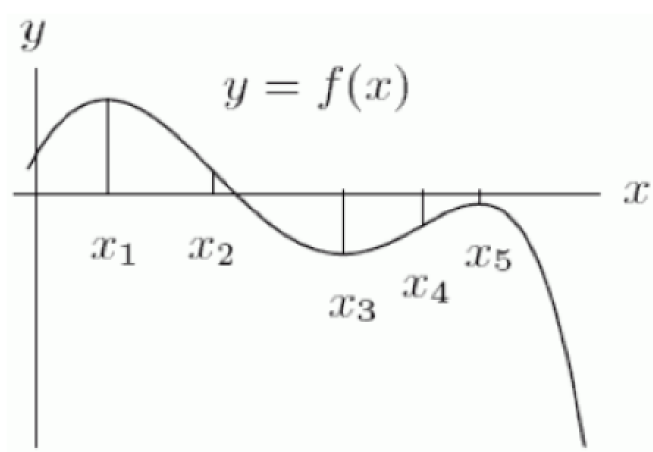
f achieves its global maximum at $x = 0$.

4. Sketch the graph of a function that has exactly two local minima, no global maximum, and one global minimum.

Here is one solution:



5. The graph of $f(x)$ is given in the following figure. What happens to $f'(x)$ at the point x_1?



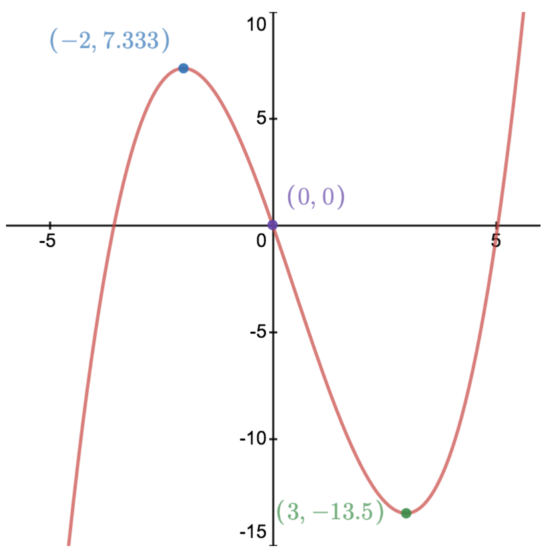
(a) $f'(x)$ has an inflection point.
(b) $f'(x)$ has a local minimum or maximum.
(c) $f'(x)$ changes sign.
(d) None of the above.

(c) $f'(x)$ changes sign.

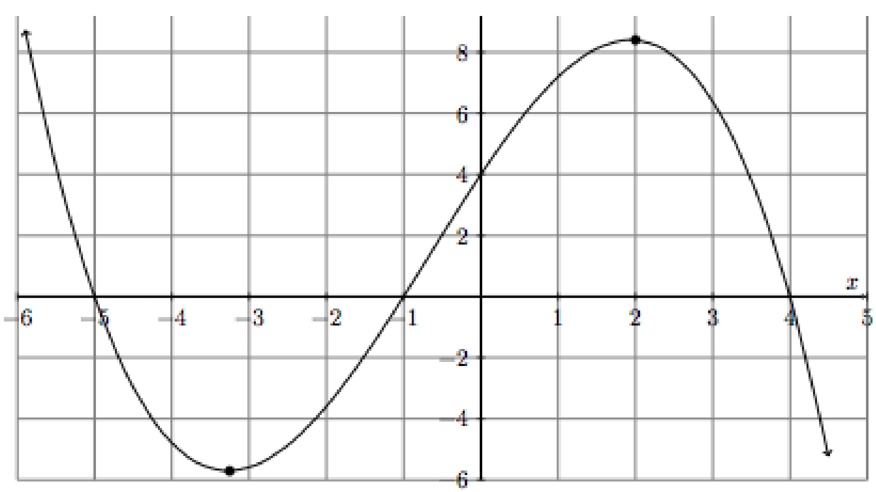
6. Sketch a smooth, continuous function satisfying all the properties listed below.

(a) $f'(x) > 0, x < -2$
(b) $f'(x) = 0, x = -2$
(c) $f'(x) < 0, -2 < x < 3$
(d) $f'(x) = 0, x = 3$
(e) $f'(x) > 0, x > 3$
(f) $f''(x) < 0, x < 0$
(g) $f''(x) = 0, x = 0$
(h) $f''(x) > 0, x > 0$

Here is one solution:



7. Consider the graph of $f'(x)$. What can be said about $f(x)$ and $f''(x)$, such as critical and inflection points, concavity, sign of $f(x)$, increasing/decreasing of $f(x)$, local maximum or minimum of $f(x)$?



- $x = -5, -1$ and 4 are critical points of f.
- $x = -3.2$ and 2 are inflection points of f.
- f is increasing when $x < -5$ and when $-1 < x < 4$, and f is decreasing otherwise.
- f is concave up when $-3.2 < x < 2$, and concave down otherwise.
- $x = -5$ and 4 correspond to local maxima of f, while $= -1$ is a local minimum.
- As $x \to \pm\infty$, the sign of f approaches $-\infty$.

8. Find the global maximum and minimum, using end behavior and derivative tests, for the function below.

   Global maximum or minimum *(circle one)* at $x =$ __-1__ is __-1/e__

$$h(x) = xe^x.$$

9. Find the global maximum and minimum for the function on the closed interval.

$$f(x) = 9x^{1/3} - 3x, \quad -1 \leq x \leq 8.$$

Global maximum at $x = 1$ is 6, global minimum at $x = -1$ and 8 is -6.

10. Find the global maximum and minimum for the function on the closed interval.

$$f(x) = x - 2\ln(x+1), \quad 0 \leq x \leq 2.$$

Global maximum at $x = 0$ is 0, global minimum at $x = 1$ is $1 - 2\ln(2) \approx -0.386$.

11. True or False problems (Explain your answers. If false, correct the statement so that it is true)

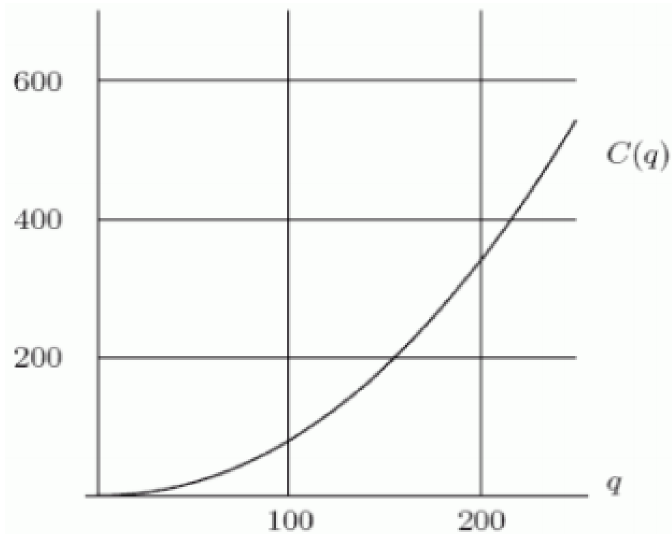
    (a) Second order derivative can be used to confirm both local and global extrema.

    False. The function $f(x) = x^4$ has a local minimum at $x = 0$, but if you plug in $x = 0$ to the second derivative $f''(x) = 12x^2$ you get 0. Thus, the second derivative test cannot confirm the local extrema. Moreover, $x = 0$ is also a global minimum for $f(x) = x^4$, so the second derivative cannot confirm global extrema either.

    (b) A function can have infinitely many derivatives.

    True. The function $f(x) = e^x$ has infinitely many derivatives.

12. The cost $C(q)$ (in dollars) of producing a quantity q of a certain product is shown in the graph below. Suppose that the manufacturer can sell the product for \$2.50 each (regardless of how many are sold), so that the total revenue from selling a quantity q is $R(q) = 2.5q$. The difference $\pi(q) = R(q) - C(q)$ is the total profit. Let q_0 be the quantity that will produce the maximum profit. What is $C'(q_0)$?



$C'(q_0) = 2.5.$

13. The revenue for selling q items is $R(q) = 600q - 2q^2$ and the total cost of producing q items is $C(q) = 120 + 60q$. What quantity maximizes profit? Find the value of the profit function at this production level.

The quantity that maximizes profit is $q = 135$. The value of the profit function when $q = 135$ is 36330.

14. Find the following limits, use l'Hopital's Rule if applicable:

(a) $\lim_{x \to \infty} \frac{6x^2 - 5x + 7}{3 + x - 3x^2 - 2x^3} = \boxed{0}$

(b) $\lim_{x \to 0} \frac{e^{x^2} - 1}{3\cos(x) - 3} = \boxed{-2/3}$

(c) $\lim_{x \to 0} \frac{e^{x^2}}{\cos(x) - 3} = \boxed{-1/2}$

(d) $\lim_{x \to 0} \frac{e^{x^2} - 1}{\cos(x) - 3} = \boxed{0}$

15. Use l'Hopital's Rule to determine which function dominates in the long run (when $x \to \infty$) $y = \ln(x + 3)$ or $y = x^{0.4}$.

$y = x^{0.4}$ dominates $y = \ln(x + 3)$.

Chapter 5. The Definite Integrals, The Fundamental Theorem of Calculus

1. Use the table to estimate $\int_0^{50} f(x)dx$ with $n = 5$ and $\Delta x = 10$ (Average left-and right-hand sums).

x	0	10	20	30	40	50
$f(x)$	40	45	55	60	80	95

Answer: 3075

2. $W(t)$ is the weight of a particular baby, in kg, at time t days after the baby's birth. $W'(t) = r(t)$ is the rate of change in the baby's weight, measured in kg/day, at time t days after the baby's birth.

(a) What are the units for

$$\int_0^{10} r(t)\ dt$$

Answer: kg

(b) What does the Fundamental Theorem of Calculus tell you about the relationship among $W(t)$, $r(t)$, $t = 0$ and $t = 10$?

Answer: $\int_0^{10} r(t)\ dt = W(10) - W(0)$

(c) The baby weighted 3.8 kg at birth and 5.8 kg at her one hundred days check up. Evaluate the definite integral

$$\int_0^{100} r(t)\ dt$$

Answer: 2 kg

3. If $E(t)$ represents the rate at which energy is consumed in a household in watts/month, t is time in months, what are the units of integral $\int_a^b E(t)dt$?
(A) $watts$ (B) $watts/month$ (C) $watts/month^2$ (D) $month$ (E) $watts^2/month^2$.
Give the practical meaning of the integral $\int_a^b E(t)dt$.

Answer: units - watts. Practical meaning - the amount of energy consumed over the time interval between a and b months.

4. A potato is cooking in an oven. Explain in words what

$$\frac{1}{35-0}\int_0^{35} f(t)dt = 300$$

means if $f(t)$ is the temperature of the potato, in degrees Fahrenheit, and t is time, in minutes.

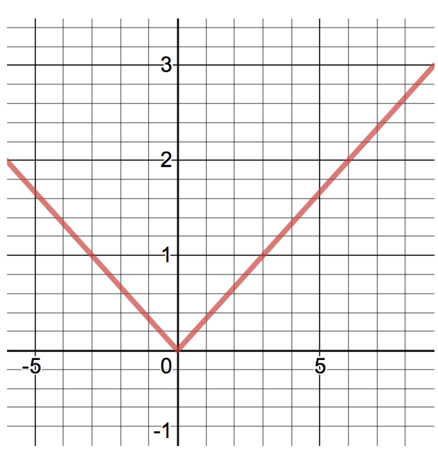
Answer: the average temperature of the potato sitting in the oven during first 35 minutes is 300 degrees Fahrenheit.

5. Find

$$\int_{-6}^{9} \frac{|x|}{3}\ dx$$

geometrically. *(Sketch the graph.)*

Answer: 19.5



6. Find the area above the line $y = 5$ and below the graph of $f(x) = 9 - x^2$

Answer: 32/3

7. True of False problems (Explain your answers. If false, correct the statement so that it is true)

   (a) If $f(x) = F'(x)$ and $\int_0^{100} f(x)dx$=-2 then $F(0) > F(100)$

   Answer: true

   (b) If $f(x) > 0$ and monotonically increasing between for $1 < x < 10$ then left hand Riemann sum of the definite integral $\int_1^{10} f(x)dx$ is greater than the actual area under the curve for $1 < x < 10$.

   Answer: false. If $f(x) > 0$ and monotonically increasing between for $1 < x < 10$ then left hand Riemann sum of the definite integral $\int_1^{10} f(x)dx$ is **less** than the actual area under the curve for $1 < x < 10$.

Chapter 6. Basic Antiderivatives

1. A factory is dumping pollutants into a lake continuously at the rate of $\frac{1}{40}t^{2/3}$ tons per week, where t is the time in weeks since the factory commenced operations. After one year of operation, how many tons of pollutant has the factory dumped into the lake? Round to 2 decimal places

   Answer: 10.87

2. Find a function F such that

$$F'(x) = x^5 - \frac{4}{\sqrt{x}}$$

   Answer:

$$\frac{x^6}{6} - 8\sqrt{x}$$

3. Find a function F such that

$$F'(x) = \sin(x) + \frac{7}{x}$$

   Answer:

$$-\cos(x) + 7\ln(|x|)$$

4. Find the indefinite integral

$$\int \left(12x^5 - 3\sec^2(x) + \frac{7}{x} - 8^x\right) dx$$

.

   Answer:

$$2x^6 - 3\tan(x) + 7\ln(|x|) - \frac{8^x}{\ln(8)} + C$$

5. Find an antiderivative $F(x)$ with $F'(x) = f(x)$ and $F(0) = 0$ for $f(x) = 9e^x - 2$.

   Answer:

$$9e^x - 2x - 9$$

6. Find the definite integral, leave exact answers:

$$\int_1^2 \left(\frac{6}{x^2}\right) dx =$$

   Answer: 3

$$\int_0^{\pi/2} [-\sin(t) - \cos(t)]\, dt =$$

Answer: -2

7. Evaluate:

$$\int \frac{x^2 - 2 \cdot x^7}{x^3} dx$$

Answer:

$$\ln(|x|) - \frac{2}{5}x^5 + C$$